Abstract—We consider a slow-fading MISO multicast channel with one $T$-antenna transmitter and $K$ single-antenna receivers with the goal of minimizing channel outage probability using quantized beamforming. Our focus is on a distributed limited feedback scenario where each receiver can only quantize and send feedback information regarding its own receiving channels.

A classical result in point-to-point quantized beamforming is that a necessary and sufficient condition for full diversity is to have $\lceil \log_2 T \rceil$ bits from the receiver. We first generalize this result to multicast beamforming systems and show that a necessary and sufficient condition to achieve full diversity for all receivers is to have $\lceil \log_2 T \rceil$ bits from each receiver. Also, for a two-receiver system and with $R$ feedback bits per receiver, we show that the outage performance with quantized beamforming is within $O(2^{-\frac{R}{T+1}})$ dBs of the performance with full channel state information at the transmitter (CSIT). This constitutes, in the context of multicast channels, the first example of a distributed limited feedback scheme whose performance can provably approach the performance with full CSIT.

Index Terms—Limited feedback, multicast, beamforming.

I. INTRODUCTION

Multicasting refers to the transmission of common information to several physically-separated receivers. In the context of physical layer, a particularly well-investigated scenario is the multiple-input single-output (MISO) multicast channel, where a $T$-antenna transmitter wishes to communicate to $K$ single-antenna receivers over fading channels [1]–[6]. In such a scenario, when channel state information (CSI) is available to the transmitter, one can maximize the overall performance (e.g., the ergodic capacity, or the outage probability) using beamforming or precoding. The capacity limits of MISO multicast channels with such CSI-adaptive transmission strategies have first been investigated in [1], where several scaling results have been derived with different assumptions on $T$ and $K$. Other work on the capacity of multicast channels in the large system limit have studied the case of correlated channels [2], and the performance of antenna subset selection [3].

Unlike a point-to-point MISO system where the optimal transmitter covariance matrix is simply a beamformer along the channel direction, closed-form expressions for the optimal covariance matrices or beamforming vectors are not known for a general MISO multicast system. Thus, there are also several studies [4]–[6] on the numerical optimization of multicast covariance matrices and beamforming vectors, with closed-form optimal solutions being available for certain values of $K$ and $T$ [5]. In particular, beamforming is optimal for $K \leq 3$, or close to optimal when $T$ is much larger than $K$ [5], [6].

Most of these previous studies assume that the transmitter has perfect knowledge of the CSI. In fact, CSI at the transmitter (CSIT) can be acquired through feedback from the receivers, each of which can acquire the knowledge of their own receiving channels through transmitter training sequences. On the other hand, since the CSI can assume any value in a multi-dimensional complex space, the assumption of perfect CSIT requires an “infinite number of feedback bits” from every receiver. In practice, each receiver can communicate only a finite number of bits per channel state as feedback information. A mathematical formulation of such a limited feedback scenario leads to a distributed quantization problem where each receiver quantizes only a part of the entire CSI.

A special case is a point-to-point MISO system with $K = 1$, where the distributed quantization problem boils down to a simple point-to-point quantization problem and several solutions are available [7]–[11]. However, very little work exists on the design of limited feedback schemes when $K > 1$. In [12], the authors study a scenario where only the channel direction information is quantized with channel magnitude information still being perfectly available at the transmitter (this would again require infinitely many receiver feedback bits). In [13], the performance of centralized (non-distributed) variable-length quantizers have been analyzed. To the best of our knowledge, the first “true” distributed quantizers for beamforming in multicast channels have been proposed in [14]. Although the quantizers in [14] can provide full diversity, they do not do so in a rate-optimal manner, and cannot provably approach the performance with full CSIT.

The rest of this paper is organized as follows. In Section II, we introduce the system model and the distributed quantizers. In Section III, we construct rate-optimal distributed quantizers that can provide full diversity. Finally, in Section IV, we design quantizers that can approach the performance with full CSIT when $K = 2$. Due to space limitations, some of the technical proofs are provided in the extended version [15] of this paper.

II. PRELIMINARIES

A. System Model

We consider a slow-fading MISO multicast channel with one transmitter with $T$ antennas and $K$ single-antenna receivers. Denote the channel from Transmitter Antenna $t$ to Receiver $k$ ($t \in \{1, \ldots, T\}$ and $k \in \{1, \ldots, K\}$) as $h_{tk} \in \mathbb{C}$. Also, let $H_k \triangleq [h_{1k} \cdots h_{Tk}]^T \in \mathbb{C}^{T \times 1}$ and $H \triangleq [H_1 \cdots H_K] \in \mathbb{C}^{T \times K}$ denote the channels from the transmitter to Receiver $k$, and the entire channel state, respectively. We assume that Receiver $k$ knows the vector $H_k$ of its own receiving channels perfectly.

Let $s \in \mathbb{C}$ denote the information-bearing symbol that we wish to multicast to the receivers. For a given channel state $H_k$, transmitters have perfect knowledge of the CSI. In fact, CSI at the
the transmitter sends \( \sqrt{P} \) over its \( T \) antennas, where \( P \) is the transmitter short-time power constraint, \( x \in \mathcal{X} \) is a beamforming vector, and \( \mathcal{X} = \{ x \in \mathbb{C}^{T \times 1} : |x| = 1 \} \) is the set of all beamforming vectors. The channel input-output relationships are \( y_k = s(h_k, x) + \eta_k, k = 1, \ldots, K \), where \( y_k \in \mathbb{C} \) and \( \eta_k \in \mathbb{C} \) are the received signal and the noise at Receiver \( k \), respectively. We assume that \( \eta_1, \ldots, \eta_K, h_{11}, \ldots, h_{TK} \) are independent circularly-symmetric complex Gaussian random variables with variance 1.

The signal-to-noise ratio (SNR) at Receiver \( k \) can be calculated to be \( |x, h_k|^2 P \). We refer to the quantity \( \gamma(x, h) = \min_k |x, h_k|^2 P \) as the “network SNR.” For a fixed \( h \) and \( x \), the capacity of the multicast channel as defined above is then \( \log_2 (1 + \gamma(x, h)) \) bits/sec/Hz. Without loss of generality, we set the target data transmission rate to be 1 bit/sec/Hz, in which case an outage occurs if \( \gamma(x, h) < 1 \).

Consider now an arbitrary mapping \( \mathcal{M} : \mathbb{C}^{T \times K} \rightarrow \mathcal{X} \), and suppose that the transmitter uses the beamforming vector \( \mathcal{M}(h) \) for a given \( h \). We define the outage probability with \( \mathcal{M} \) as \( \text{out}(\mathcal{M}) = P(\mathcal{M}(h), h) < 1 \). We also let \( \text{d}(\mathcal{M}) = \lim_{P \to \infty} \frac{\log \text{out}(\mathcal{M})}{\log P} \) denote the diversity gain with \( \mathcal{M} \).

If the transmitter somehow knows the entire channel state \( h \) perfectly, we have a “full-CSIT system.” In such a scenario, given \( h \), the transmitter can use an optimal beamforming vector, say \( F(h) \), so as to minimize the outage probability. Clearly, the minimum-possible outage probability can be reached by maximizing the network SNR for each channel state. Hence, we define the corresponding full-CSIT mapping \( F \) as

\[
F(h) \triangleq \arg \max_{x \in \mathcal{X}} \gamma(x, h) = \arg \max_{x \in \mathcal{X}} \min_k |(x, h_k)|^2,
\]

with ties broken arbitrarily. Note that the calculation of the optimal beamforming vector \( F(h) \) requires the knowledge of the entire channel state \( h \). However, none of the terminals can acquire \( h \) in its entirety. In fact, Receiver \( k \) can only acquire its own local channel states \( h_k \) via transmitter training. To calculate \( F(h) \), the \( K \) parts \( h_1, \ldots, h_K \) of the channel state \( h \) should be available to the transmitter, which would require an “infinite rate of feedback” from all of the \( K \) receivers.

Hence, suppose Receiver \( k \) can send only \( b_k \) bits of feedback at each channel state. We formalize such a distributed limited feedback scenario via a distributed quantizer \( Q : \mathbb{C}^{T \times K} \rightarrow \mathcal{X} \), which is specified by \( K \) encoders \( E_k : \mathbb{C}^{T \times 1} \rightarrow B_{b_k}, k = 1, \ldots, K \), and a decoder \( D : \prod_{k=1}^K B_{b_k} \rightarrow \mathcal{X} \), where \( B_{b_n} \) is the set of all binary codewords of length \( n \). The \( k \)th encoder \( E_k \) is available at Receiver \( k \), and the decoder \( D \) is available at all the terminals. The feedback transmission scheme then operates as follows: Given \( h \), Receiver \( k \) feeds back \( E_k(h_k) \) as multicast to all terminals. We assume that each terminal receives the \( K \) feedback messages \( E_1(h_1), \ldots, E_K(h_K) \) without any errors or delays to reproduce the quantized beamforming vector \( \tilde{Q}(h) \triangleq D(E_1(h_1), \ldots, E_K(h_K)) \). We refer to the set \( \{ Q(h) : h \in \mathbb{C}^{T \times K} \} \) as the quantizer codebook. The outage probability \( \text{out}(Q) = P(\gamma(Q(h), h) < 1) \) is thus achieved with a feedback rate of \( R = \log_2 Q(h) \) bits per channel state at Receiver \( k \).

The adjective “distributed” signifies that there are potentially many \( (K > 1) \) non-communicating quantizer encoders each of which quantizes only a part of the entire CSI. Note that in the case \( K = 1 \), we have a “non-distributed” quantizer with a single quantizer encoder at the sole receiver and a quantizer decoder at the transmitter. To gain initial insight on the problem of designing distributed quantizers, we discuss the existing non-distributed quantizer design methodology for \( K = 1 \), and show why the same design ideas cannot immediately be applied to the case of \( K > 1 \).

### B. Non-Distributed vs. Distributed Quantization

For a given codebook \( C \subset \mathcal{X} \) of beamforming vectors, let \( M^*_C(h) \triangleq \arg \max_{x \in C} \gamma(x, h) \).

It can be shown that \( M^*_C(h) \) is an optimal mapping for codebook \( C \) in the sense that for any other mapping \( \mathcal{M} : \mathbb{C}^{T \times K} \rightarrow C \), we have \( \text{out}(M^*_C) \leq \text{out}(\mathcal{M}) \).

In a point-to-point system \( (K = 1) \), the mapping \( (2) \) can easily be realized with limited feedback: The sole receiver feeds back \( \log_2 |C| \) bits that can uniquely represent the SNR-maximizing beamforming vector \( M^*_C(h) \) in \( C \). Using these feedback bits, the transmitter recovers and transmits via \( M^*_C(h) \). Therefore, when \( K = 1 \), it is clear how to optimally design the quantizer encoding and decoding functions for a given codebook. The problem of designing a good quantizer boils down to the design of good codebooks, and several constructions (e.g., Grassmannian codebooks) are available.

However, in a multicast system with more than one receiver \( (K > 1) \), none of the receivers can, by itself, determine the beamforming vector \( M^*_C(h) \) that provides the highest network SNR. This is because the network SNR \( \gamma(Q(h), h) \) provided by a given beamforming vector \( h \in C \), depends in general on all the \( KT \) channels from the transmitter to the \( K \) receivers. Therefore, when \( K > 1 \), for a general codebook \( C \), it is not immediately clear how to implement the optimal mapping in \( (2) \), or whether such an implementation is even possible.

The absence of a rate-limited distributed implementation of \( (2) \) is a fundamental difficulty in designing structured distributed quantizers. Our general quantizer design strategy is thus to forget about picking the best beamforming vector, and instead focus on not picking the worst beamforming vector(s) from the set of all binary codewords of length \( n \). The \( k \)th encoder \( E_k \) is available at Receiver \( k \), and the decoder \( D \) is available at all the terminals. The feedback transmission scheme then operates as follows: Given \( h \), Receiver \( k \) feeds back \( E_k(h_k) \) as multicast to all terminals. We assume that each terminal receives the \( K \) feedback messages \( E_1(h_1), \ldots, E_K(h_K) \) without any errors or delays to reproduce the quantized beamforming vector \( \tilde{Q}(h) \triangleq D(E_1(h_1), \ldots, E_K(h_K)) \). We refer to the set \( \{ Q(h) : h \in \mathbb{C}^{T \times K} \} \) as the quantizer codebook. The outage probability \( \text{out}(Q) = P(\gamma(Q(h), h) < 1) \) is thus achieved with a feedback rate of \( R \) bits per channel state at Receiver \( k \).

The adjective “distributed” signifies that there are potentially many \( (K > 1) \) non-communicating quantizer encoders each of which quantizes only a part of the entire CSI. Note that in the case \( K = 1 \), we have a “non-distributed” quantizer with a single quantizer encoder at the sole receiver and a quantizer decoder at the transmitter. To gain initial insight on the problem of designing distributed quantizers, we discuss the existing non-distributed quantizer design methodology for \( K = 1 \), and show why the same design ideas cannot immediately be applied to the case of \( K > 1 \).

### III. DIVERSITY GAINS OF DISTRIBUTED QUANTIZERS

In this section, we design distributed quantizers that can achieve the full-CSIT diversity gain \( T \). For a point-to-point
MISO system with beamforming, it is well-known [8] that a necessary and sufficient condition to achieve full diversity is to have \( \lceil \log_2 T \rceil \) feedback bits from the receiver. Here, we generalize this result to the multicast setting by showing that a necessary and sufficient condition to achieve the full-diversity gain \( T \) is to have \( \lceil \log_2 T \rceil \) feedback bits from every receiver. We first consider the achievability. In order to design a distributed quantizer that achieves full diversity, we recall our general design strategy in Section II-B: Instead of trying to pick the best beamforming vector in a given codebook, we instead focus on not picking the worst beamforming vector(s) in a given codebook. For this purpose, for any given \( n \geq T \), let \( C_n \triangleq \{ x_1, \ldots, x_n \} \subset X \) be an arbitrary set of beamforming vectors such that for any \( n \geq T \), any \( T \) of the vectors in \( C_n \) (chosen without repetition) are linearly independent.\(^1\)

Consider now a MISO system where \( K = 1 \). For any given \( h \in C^T \times X \) and \( n \geq T \), the vectors in \( C_n \) can be ordered from the worst to the best in terms of the SNR provided by each. In other words, we have \( |\langle x_i, h_1 \rangle|^2 P \leq \cdots \leq |\langle x_i, h_1 \rangle|^2 P \) for some \( i_1, \ldots, i_n \) with \( \{ i_1, \ldots, i_n \} = \{ 1, \ldots, n \} \). Note that the ordering indices \( i_1, \ldots, i_n \) depend on the channel state \( h = h_1 \). We consider the mapping \( \mathcal{W}_n (h_1) \triangleq x_{i_T} \) that chooses the \( \lceil T/2 \rceil \)th worst beamforming vector in \( C_n \). The following proposition, whose proof can be found in [15], shows that \( \mathcal{W}_n \) provides full diversity.

**Proposition 1.** Let \( K = 1 \). Then, \( d(\mathcal{W}_n) = T \), \( \forall n \geq T \).

Hence, in the MISO setting, it is not necessary to choose the best beamforming vector in a codebook to achieve full diversity. One just has to avoid the \( T - 1 \) worst beamforming vectors and pick at least the \( T \)th worst vector in the given codebook. We now show how this observation can be applied to the multicast setting for designing a distributed quantizer that can achieve full diversity. We first provide an example for \( T = 4 \), \( K = 2 \), and then state and prove the general case.

**Example 1.** Let \( T = 4 \) and \( K = 2 \). We design a distributed quantizer that achieves diversity by using \( 2 \)-bit feedback bits per receiver per channel state. Consider the codebook \( C_{16} = \{ x_1, \ldots, x_{16} \} \). Note that any \( T = 4 \) of the 16 vectors in \( C_{16} \) are linearly independent. We imagine the vectors in \( C_{16} \) as cells of a \( 4 \times 4 \) grid as shown in Fig. 1. In a sense that is to be made precise in the following, Receiver 1 will be "working on" the columns of the grid, while Receiver 2 will work on the rows of the grid. Each row/column is uniquely represented by one of the \( 2 \)-bit binary codewords. All the data in Fig. 1 will be available at both receivers and the transmitter.

Consider now a distributed quantizer, namely \( \mathcal{Q} \), that operates as follows: Given channel state \( h = [h_1, h_2] \), Receiver 1 calculates and sorts its SNR values as \( |\langle x_i, h_1 \rangle|^2 P \leq \cdots \leq |\langle x_i, h_1 \rangle|^2 P \) for some \( i_1, \ldots, i_{16} \) that do not contain any one of the \( 3 \) worst beamforming vectors \( x_{i_1}, x_{i_2}, x_{i_3} \). Receiver 1 feeds back the \( 2 \)-bit binary codeword that represents \( I_r \). (For example, if \( i_1 = 9, i_2 = 7 \) and \( i_3 = 16 \), we have \( I_r = 2 \), and Receiver 1 feeds back \( 01 \)). Similarly, Receiver 2 calculates and sorts its SNR values as \( |\langle x_j, h_2 \rangle|^2 P \leq \cdots \leq |\langle x_j, h_2 \rangle|^2 P \) for some \( j_1, \ldots, j_{16} \) that do not contain any one of the \( 3 \) worst beamforming vectors \( x_{j_1}, x_{j_2}, x_{j_3} \) for Receiver 2. Receiver 2 feeds back 2 bits that represents \( I_r \). The transmitter recovers the indices \( I_r \) and \( I_c \), and transmits over the beamforming vector in the \( I_r \)th row, \( I_c \)th column of the grid in Fig. 1.

Now, using a union bound over all receivers, we have

\[
\begin{align*}
\text{out}(\mathcal{Q}) &= P \left( \min_{k \in \{1,2\}} |\langle \mathcal{Q}(h), h_k \rangle|^2 P < 1 \right) \\
&\leq \sum_{k=1}^2 P \left( |\langle \mathcal{Q}(h), h_k \rangle|^2 P < 1 \right).
\end{align*}
\]

On the other hand, by construction, the quantizer \( \mathcal{Q} \) avoids any of the \( 3 \) worst beamforming vectors for any of the receivers. Hence, by Proposition 1, for any \( k \in \{1,2\} \), we have \( P(|\langle \mathcal{Q}(h), h_k \rangle|^2 P < 1) \in O(P^{-4}) \). This implies \( d(\mathcal{Q}) = 4 \).

The construction in Example 1 extends to the case of an arbitrary number of receivers and transmitter antennas in a straightforward manner. The final result is summarized by the following proposition, whose proof is provided in [15].

**Proposition 2.** For any \( T \) and \( K \), there is a quantizer \( \mathcal{Q} \) with \( d(\mathcal{Q}) = T \) and \( R_k(\mathcal{Q}) = \lceil \log_2 T \rceil \), \( \forall k \).

We also prove the following converse result in [15].

**Proposition 3.** For any quantizer \( \mathcal{Q} \) with \( R_k(\mathcal{Q}) < \lceil \log_2 T \rceil \) for some \( k \in \{1,\ldots,K\} \), we have \( d(\mathcal{Q}) < T \).

The main result of this section is then the following combined restatement of Propositions 2 and 3.

**Theorem 1.** A necessary and sufficient condition to achieve the full diversity gain in a quantized multicast beamforming system is to have \( \lceil \log_2 T \rceil \) feedback bits from each receiver.

\[\text{Fig. 1: An example quantizer for } T = 4, K = 2.\]
This generalizes the classical result for point-to-point MISO systems to multicast systems.

IV. APPROACHING THE FULL-CSIT PERFORMANCE WITH DISTRIBUTED QUANTIZATION

We now consider the design of distributed quantizers whose outage probabilities can be made arbitrarily close to that of a full-CSIT system. This will be accomplished in two steps. As the first step, in Section IV-A, we show that for any arbitrary codebook $C^*$, one can synthesize a distributed quantizer that can achieve the same performance as the centralized quantizer $Q^*_C$. Then, as the second step, in Section IV-B, we show that the outage probabilities of centralized quantizers with well-designed codebooks can approach the full-CSIT outage probability. A combined restatement of our results in these two steps will show the existence of distributed quantizers whose performances can approach that of a full-CSIT system.

A. The Synthesis of a Distributed Quantizer out of a Centralized Quantizer

As we have mentioned in Section II-B, the main difficulty in designing distributed quantizers is the absence of a rate-limited distributed implementation of the optimal centralized quantizer $Q^*_C(h) = \arg \max_{x \in \mathcal{C}} \min_k |\langle x, h_k \rangle|^2$ for a given codebook $C$. If this difficulty could be overcome, the problem of designing a good distributed quantizer would boil down to the much easier problem of designing a good quantizer codebook.

Fortunately, for the outage probability performance measure, we do not need to implement $Q^*_C$ as it is. In fact, an outage event with $Q^*_C$, i.e. the event $\max_{x \in \mathcal{C}} \min_k |\langle x, h_k \rangle|^2 P < 1$, occurs if and only if $\max_{x \in \mathcal{C}} \min_k 1(\langle x, h_k \rangle)^2 P < 1 = 1$ (For a logical statement $S$, we let $1(S) = 1$ if $S$ is true, and otherwise, we let $1(S) = 0$). Hence, for 

$$ Q_C(h) \triangleq \arg \max_{x \in \mathcal{C}} \min_k 1(\langle x, h_k \rangle)^2 P < 1 \quad (5) $$

(with ties broken arbitrarily), we have $\text{out}(Q_C) = \text{out}(Q^*_C)$. But now, in contrast to the mapping $Q^*_C$, the new mapping $Q_C$ can be realized as a distributed quantizer. In fact, Receiver $k$ can calculate the $|C|$ binary values $1(\langle x, h_k \rangle)^2 P < 1$, $x \in \mathcal{C}$ and feed them back using $|C|$ feedback bits. The transmitter can then determine $Q_C(h)$ for every given $h$ via $|C|$ feedback bits from each receiver. Hence, for any codebook $C$, we have $\text{out}(Q_C) = \text{out}(Q^*_C)$ with $R_k(Q_C) = |C|, \forall k$.

There is one particular disadvantage of our construction so far: In general, the synthesis of the distributed version of an $R$-bit centralized quantizer requires $2^R$ bits per receiver. In order to resolve this exponential rate amplification problem, we revisit the operation of the encoders of $Q^*_C$ at the receivers. Consider, for example, the quantizer encoding operation at Receiver 1. For a given codebook $C = \{x_i : i = 1, \ldots, |C|\}$, what Receiver 1 feeds back can be thought as a configuration $\{i : |\langle x_i, h_1 \rangle|^2 P \leq 1\}$, i.e. a set of beamforming vectors in $\mathcal{C}$ that result in outage at Receiver 1 given that the channel state from the transmitter to Receiver 1 is $h_1$. Now, let 

$$ \chi(C) \triangleq |\{i : |\langle x_i, h_1 \rangle|^2 P \leq 1\} : h_1 \in \mathcal{C}^{T \times 1}| \quad (6) $$

denote the cardinality of the collection of all configurations given $C$. In order to convey the binary values $1(\langle x, h_k \rangle)^2 P < 1$, $x \in C$ to the transmitter, it is then sufficient for each receiver to send $|\log_2 \chi(C)|$ feedback bits for every channel state. We can calculate the 

$$ \chi(C) = \min \left\{ |C|, 16 \left[ \sum_{i=0}^{2^T} (|C|-1)^4 \right] \right\}. \quad (7) $$

Note that the upper bound in (7) is $O(|C|^{8T})$ as $|C| \to \infty$. Therefore, for any codebook $C$, the binary values $1(\langle x, h_k \rangle)^2 P < 1$, $x \in C$ can be losslessly conveyed from Receiver $k$ to the transmitter using $8T \log_2 |C| + O(1)$ bits. Using these feedback bits, the transmitter can determine $Q_C(h)$ in the same manner as discussed in the beginning of this section. The resulting quantizer, which we shall refer to as $Q^*_C$ from now on, achieves $\text{out}(Q_C) = \text{out}(Q^*_C) = \text{out}(Q^*_C)$. This establishes the following main result of this section.

Theorem 4. For any codebook $C$, we have $\text{out}(Q_C) = \text{out}(Q^*_C)$ with $R_k(Q_C) = 8T \log_2 |C| + O(1), \forall k$.

Hence, for any rate-$R$ centralized quantizer, we can synthesize a distributed quantizer that achieves the same performance as the centralized quantizer and can operate with roughly $8TR$ bits per receiver.

B. The Existence of Good Centralized Quantizer Codebooks

In Section IV-A, we have synthesized distributed quantizers out of centralized quantizers in a rate-efficient manner. To design a good distributed quantizer whose performance can approach the full-CSIT performance, it is now sufficient to design a good centralized quantizer. Since the performance of an optimal centralized quantizer depends only on its codebook, we just need to design a good quantizer codebook. Here, we utilize “δ-covering codebooks” as defined in the following.

Definition 1. Let $\delta \in (0, 1)$. We call $D_\delta$ a $\delta$-covering codebook if $\forall x \in X$, $\exists y \in D$ such that $|\langle y, x \rangle|^2 \geq 1 - \delta$.


Example 2. Given $\delta \in (0, 1)$, let $s_\delta \triangleq 2^{\lceil \log_2(2T/\delta) \rceil} + 1$, $S_\delta \triangleq \{-1 + ks_\delta, k = 0, \ldots, 2s_\delta^{-1}\}$, and $Y_\delta \triangleq \{y/|y| : \Re(e^{jy_i}), \Im(e^{jy_i}) \in S_\delta, \forall i, 0 < |y| \leq 1\}$. According to [11, Proposition 3], for any $\delta \in (0, 1)$, the codebook $Y_\delta$ is a $\delta$-covering codebook with $|Y_\delta| \in O(\delta^{-2T})$.

Let us now discuss how to make use of $\delta$-covering codebooks in a point-to-point MISO system with $K = 1$. Consider the optimal quantizer $M^*_C(h)$ for codebook $D_\delta$. Note that $M^*_C$ can be implemented using $\log_2 |D_\delta|$ feedback bits. By definition, we have $\forall h_1, \exists y \in D_\delta, |\langle h_1, y \rangle|^2 \geq
1 − δ. This leads to the lower bound |⟨\mathcal{M}_B^{\star}(h_1), h_1⟩|^2 P ≥ |\mathcal{F}(h_1), h_1⟩|^2 P(1 − δ) = \|h_1\|^2 P(1 − δ). Therefore, a well-designed quantizer can provide a uniformly bounded multiplicative SNR loss. This is a very useful property as most performance measures (such as capacity or outage probability) are monotonic functions of the SNR. We can thus conclude that the performance of a rate-[\log_2|\mathcal{D}_B|] quantized beamforming system at power \( P \) is at least that of a full-CSIT system at power \( P(1 − δ) \). In particular, for the codebook \( \mathcal{Y}_B \), the performance with \( R \) bits of feedback at power \( P \) is no worse than the full-CSIT performance at power \( P(1 − O(2^{−δ})) \).

Then, a fundamental question is to determine whether or not the SNR loss due to quantization can similarly be uniformly bounded for a general multicast system with \( K > 1 \) receivers. The positive answer is provided by the following theorem for the special case of \( K = 2 \) receivers. The proof of the theorem can be found in [15].

**Theorem 3.** Let \( K = 2 \). Then, for every \( h \in \mathbb{C}^{T \times K} \), there exists \( y \in \mathcal{Y}_B \) such that \( \gamma(y, h) \geq \gamma(\mathcal{F}(h), h)(1 − O(\sqrt{\delta})) \).

Hence, for a two-receiver system, for any channel state, one can quantize the full-CSIT beamforming vector \( \mathcal{F}(h) \) to a quantized vector \( y \in \mathcal{D}_B \) such that the SNR loss is always within \((1 − O(\sqrt{\delta}))\) of the SNR with full CSIT.

Now, let \( \text{out}(\mathcal{M}; P) \) denote the outage probability of mapping \( \mathcal{M} \) for a given power constraint \( P \). Combining Theorems 2 and 3, we obtain the following main result for a two-user multicast system with distributed quantized beamforming.

**Theorem 4.** Let \( K = 2 \). With \( R \) feedback bits per receiver per channel state, an outage performance of \( \text{out}(\mathcal{F}; P(1 − O(2^{−\frac{R}{2T^2}π})) \) is achievable at any \( P \).

**Proof.** By Theorem 3, we have \( \text{out}(\mathcal{M}_B^{\star}; P) \leq \text{out}(\mathcal{F}; P(1 − O(\sqrt{\delta}))) \). By Theorem 2, \( \mathcal{M}_B^{\star} \) can be realized as the distributed quantizer \( \mathcal{Q}_B \) using \( ST\log_2|\mathcal{D}_B| + O(1) = 16T^2\log_2\frac{1}{δ} + O(1) \) bits per each receiver. The equality follows as \( |\mathcal{Y}_B| \in O(\delta^{−2T}) \) as shown in Example 2. Setting \( R = 16T^2\log_2\frac{1}{δ} + O(1) \) and solving for \( δ \), we obtain the statement of the theorem. \( \square \)

One way to visualize the outage probability loss due to quantization is that in the usual graph of \( P \) in the horizontal axis versus the outage probability in the vertical axis (where both axes are in the logarithmic scale), the outage probability with \( R \) bits of feedback per receiver is at most the full-CSIT curve shifted \(-10\log_{10}(1 − O(2^{−\frac{R}{2T^2}π})) \) dBs to the right. Equivalently, since as \( x \to 0, −\log(1 − O(x)) = O(x) \), the outage probability with \( R \) bits of feedback is within \( O(2^{−\frac{R}{2T^2}π}) \) dBs to the outage probability with full CSIT.

For the case \( K = 1 \), it is known (see e.g. [9]) that the performance with quantized beamforming is at most within \( O(2^{−\frac{R}{2T^2}π}) \) dBs to the outage probability with full CSIT. Hence, despite the complicated distributed nature of the quantizer design problem for \( K = 2 \), the performance loss due to quantization can still be made to decay exponentially with the per-receiver feedback rate \( R \) as \( O(2^{−\frac{R}{2T^2}π}) \). The factor \( 32T^2 \) is likely not the best possible, and can perhaps be improved (made smaller) with more work.

For \( K ≥ 3 \), our results are not strong enough to prove that one can uniformly approach the full-CSIT performance using distributed feedback. The difficulty is to show the existence of good centralized quantizers whose performances can uniformly approach the full-CSIT performance (This can be accomplished by an extension of Theorem 3 to \( K ≥ 3 \)). Nevertheless, if such a result had been available, it would have been straightforward to synthesize good distributed quantizers out of the good centralized quantizers via the same arguments as in Section IV-A. In this context, intuition suggests, without much room for doubt, that a sequence of optimal centralized quantizers for a sequence of \( δ \)-covering codebooks should uniformly approach the full-CSIT performance as \( δ \to 0 \). A formal proof, however, remains elusive.

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**REFERENCES**