SUBJECTIVE BELIEFS AND MULTIPLE PRIORS WITHOUT TRANSITIVITY AND MONOTONICITY

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Abstract

I relax transitivity and monotonicity in the subjective expected utility and multiple priors models. My extensions accommodate various behavioral patterns, such as partition dependence, source dependence, comparative ignorance, and preference reversals.

I model complete preferences over Anscombe–Aumann’s state-contingent acts. My main results characterize bi-partitional representations where any acts $f$ and $g$ are compared via probabilistic beliefs or sets of beliefs that depend on the partitions $\pi$ and $\tau$ generated by $f$ and $g$ respectively. The beliefs $p(\pi, \tau)$ and sets of beliefs $M(\pi, \tau)$ are determined uniquely on any generated partition $\pi$. Monotonicity and transitivity are not assumed, but can be imposed separately and interpreted in terms of simple mathematical properties of the functions $p$ and $M$. Various refinements and applications are discussed.

1 Introduction

Transitivity and monotonicity are required in the classic subjective expected utility (SEU) models of Savage [30] and Anscombe and Aumann [3] (henceforth AA). Both conditions are maintained in various extensions of SEU, including

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well-known utility representations for *ambiguity*: the multiple priors model (Gilboa and Schmeidler [17], henceforth GS), Choquet expected utility (Schmeidler [31]), variational preferences (Maccheroni, Marinacci, and Rustichini [27]), and others,

- the Bewley-type representations for incomplete preferences that reflect indecisiveness (Bewley [4], Ok, Ortoleva, and Riella [29]), objective rationality (Gilboa, Maccheroni, Marinacci and Schmeidler [16]), choice deferral (Kopylov [24]) and other interpretations.

Thus it is common to impose transitivity and monotonicity even when other standard axioms—such as completeness, AA’s Independence, or Savage’s Sure-Thing Principle—are relaxed.

Yet there is a growing evidence that choices among state-contingent prospects can be non-monotonic, intransitive, or both. Tversky and Kahneman [36] (henceforth TK) attribute some non-monotonic patterns to *representativeness heuristics* where subjective confidence in a proposition $E$ can be increased by additional constraints that are salient or representative for $E$. For example, a majority of TK’s subjects choose to bet on a random sequence $RGRRRG$ rather than on $RGRRR$ when the outcome $G$ is known to be twice as likely as its alternative $R$. In other words, they exhibit the strict ranking $f \succ g$ for acts

$$f = \begin{cases} $10 & \text{if } RGRRRG \text{ occurs} \\ $0 & \text{otherwise} \end{cases} \quad \text{and} \quad g = \begin{cases} $10 & \text{if } RGRRR \text{ occurs} \\ $0 & \text{otherwise} \end{cases}$$

(1)

even though $g(\omega) \geq f(\omega)$ for all possible states $\omega$. Here the longer sequence $RGRRRG$ can appear more representative than $RGRRR$ because it increases the proportion of the more likely outcomes $G$. Similarly, TK’s *availability heuristics* can produce non-monotonic beliefs (such as the “Linda effect”) where events $E$ are deemed more likely than $A \supset E$ if it is cognitively easier to produce examples that support $E$.

Next, Birnbaum’s [5] critique of the first-order stochastic dominance for preferences over objective lotteries can be translated for state-contingent acts. Consider a state space $\Omega = \{1, 2, \ldots, 100\}$ and two prospects $f$ and $g$ such that

$$f = \begin{cases} $96 & \text{if } 1 \leq \omega \leq 85 \\ $90 & \text{if } 86 \leq \omega \leq 90 \\ $12 & \text{if } 91 \leq \omega \leq 100 \end{cases} \quad \text{and} \quad g = \begin{cases} $96 & \text{if } 1 \leq \omega \leq 90 \\ $14 & \text{if } 91 \leq \omega \leq 95 \\ $12 & \text{if } 96 \leq \omega \leq 100. \end{cases}$$

(2)

Birnbaum and Navarette [6] observe that 70% of subjects choose $f$ over $g$ when the state space $\Omega$ is specified as a composition of 100 balls in an urn. Thus the ranking $f \succ g$ is plausible even though $g(\omega) \geq f(\omega)$ in each state $\omega \in \Omega$. One possible explanation is the prevalence of high payoffs in the range $f(\Omega) = \{$$96, $90, $12\}$ and low payoffs in the range $g(\Omega) = \{$$96, $14, $12\}$. 

2
Intransitive choices under uncertainty can take the form of preference reversals where the ranking \( f \succ g \) is inconsistent with the cash equivalents \( C_f < C_g \) that are elicited separately for \( f \) and \( g \):

\[
f \succ g \sim C_g \succ f.
\]


Some preference reversals can be derived from other well-established behavioral patterns, such as partition dependence. For example, Sonnemann, Camerer, Fox, and Langer [33] set up experimental markets for contingent assets

\[
t(A) = \begin{cases} 
$1 & \text{if } \xi \in A \\
$0 & \text{if } \xi \notin A 
\end{cases}
\]

where \( \xi \in \mathbb{R} \) is a random variable (like the DAX index closing value), and \( A \subset \mathbb{R} \) is an interval on the real line. In the first market, the tickets \( t(A) \) are traded together with two other assets \( t(B_1) \) and \( t(C_1) \) such that \( B_1 \) and \( C_1 \) partition \( A \). In the second market, \( t(A) \) are traded together with \( t(B_2) \) and \( t(C_2) \) such that \( B_2 \) and \( C_2 \) partitioned the complement of \( A \). The average price of the ticket \( t(A) \) is observed to be substantially higher (by about 20-25 cents) in the former market than in the latter. Consider two portfolios

\[
f = 100t(A) \quad \text{and} \quad g = 100t(B_1) + 90t(C_1)
\]

that consist of the specified amounts of the tickets \( t(A) \), \( t(B_1) \), and \( t(C_1) \) such that \( B_1 \) and \( C_1 \) partition \( A \). The experiments of Sonnemann et al. suggest that subjects can be willing to pay more for the portfolio \( g \) than for \( f \). These subjects must either violate monotonicity \( g \succ f \) or allow preference reversals \( f \succeq g \sim C_g \succ f \).

Another relevant empirical pattern is comparative ignorance. Fox and Tversky [14] observe that “ambiguity aversion will be present when subjects evaluate clear and vague prospects jointly, but it will greatly diminish or disappear when they evaluate each prospect in isolation.” In one experiment, UC Berkeley students evaluated prospects \( f_I \) and \( f_{SF} \) paying $100 contingent on the two events \( I \) and \( SF \) when the afternoon high temperature on a given day exceeds 60F in Istanbul and in San Francisco respectively. The average cash equivalent for the prospects \( f_I \) was $38 when \( h_I \) was priced on its own, but only $24 when \( f_I \) was priced together with \( f_{SF} \). Consider two portfolios

\[
f = \$30 + t(SF) \quad \text{and} \quad g = f_I = 100t(I)
\]

where the unitary tickets \( t(SF) \) and \( t(I) \) are contingent on the events \( SF \) and \( I \) respectively. The ranking \( f \succ g \) is plausible in Fox and Tversky’s experiments.
because the evaluation of \( g \) in the presence of bets on San Francisco weather should be less than $30, which is explicitly guaranteed by \( f \). However, the average \( C_g \) is $38 and hence, should exceed $31 and a fortiori, \( C_f \).

### 1.1 Representations and Results

To accommodate diverse non-monotonic and intransitive behaviors, I extend AA’s subjective expected utility (SEU) and GS’s maxmin expected utility representations (MEU, also called the multiple priors model). I maintain AA’s basic decision framework, where preferences \( \succeq \) are defined over uncertain prospects (acts) that map a finite state space \( \Omega \) into a simplex of lotteries \( X \). The SEU and MEU representations are respectively

\[
U(f) = u(f(p)) \quad \text{and} \quad U(f) = \min_{q \in M} u(f(q))
\]

where \( u \) is an expected utility index on \( X \), \( p \) is a probability measure (belief) on \( \Omega \), \( M \) is a convex and closed set of beliefs, and the mixture \( f(q) = \sum_{\omega \in \Omega} q(\omega) f(\omega) \) is defined for any act \( f \) and any belief \( q \).

Consider the following representation:

\[
f \succeq g \iff u(f(p(\pi_f, \pi_g))) \geq u(g(p(\pi_g, \pi_f)))
\]

(5)

where any pair of prospects \( f \) and \( g \) is compared via subjective beliefs \( p(\pi_f, \pi_g) \) and \( p(\pi_g, \pi_f) \) that depend on the partitions \( \pi_f \) and \( \pi_g \) generated by \( f \) and \( g \) respectively. Call this model a bi-partitional expected utility (BPEU) representation.

Similarly, define bi-partitional maxmin expected utility (BPMEU) representations for all \( f \) and \( g \):

\[
f \succeq g \iff \min_{q \in M(\pi_f, \pi_g)} u(f(q)) \geq \min_{q \in M(\pi_g, \pi_f)} u(g(q)).
\]

(6)

The decision maker as portrayed by (6) compares any pair of acts \( f \) and \( g \) via the least favorable probabilistic scenarios selected from the convex and closed sets \( M(\pi_f, \pi_g) \) and \( M(\pi_g, \pi_f) \) respectively. These sets depend on the generated partitions \( \pi_f \) and \( \pi_g \).

The BPEU and BPMEU models have several interesting features.

- All beliefs \( p(\pi, \tau) \) and sets \( M(\pi, \tau) \) are determined uniquely on the partition \( \pi \) and hence, provide a unique evaluation for any prospect \( f \) that generates \( \pi \) in a comparison with any other \( g \).
- Preferences \( \succeq \) that are represented by (5) or (6) must be complete, but need not be transitive or monotonic. In particular, the BPEU model (5) captures non-monotonic rankings as in (1) or (2) because the beliefs \( p(\pi_f, \pi_g) \)
and \( p(\pi g, \pi f) \) can be distinct when \( \pi f \neq \pi g \). Preference reversals can be explained by the differences in the beliefs \( p(\pi f, \pi g) \) and \( p(\pi f, \pi x) \) that are used to compare \( f \) with another uncertain prospect \( g \) and with a constant \( x \). Moreover, \( \succeq \) can violate mixture continuity and Independence.

- Monotonicity and transitivity can be interpreted in terms of simple mathematical properties of the functions \( p(\cdot) \) and \( M(\cdot) \). Transitivity delivers the univariate structure \( p(\pi, \tau) = p(\pi, \pi) \) and \( M(\pi, \tau) = M(\pi, \pi) \), while monotonicity is equivalent to the symmetry \( p(\pi, \tau) = p(\tau, \pi) \) and \( M(\pi, \tau) = M(\tau, \pi) \).

- Both BPMEU and BPEU representations can be refined to a cross-partitional form and then extended to non-binary choices among state-contingent prospects.

- The beliefs \( p(\pi, \tau) \) can vary together with \( \pi \) and \( \tau \), but they must be preserved on endogenously defined separable partitions. These partitions can be applied to identify comparative ignorance in a fully subjective way. The sets of priors \( M(\pi, \tau) \) are preserved on another endogenous class of resolute partitions.

My main representation results (Theorems 2 and 3) characterize the BPMEU and BPEU models (6) or (5) respectively by relaxing GS’s list of axioms. Note that BPEU is derived as a special case of BPMEU via a version of Dekel’s [8] Betweenness axiom. Theorem 4 obtains cross-partitional refinements. Theorem 5 characterizes separable and resolute partitions. Examples and applications are discussed in Section 4.

1.2 Related Literature

Several strands of literature relax monotonicity and transitivity for choices under uncertainty. First, monotonicity is dropped when subjective beliefs are identified for state-dependent utilities (as in Karni [21]). However, state dependence is implausible in examples like (1)–(4) that have monetary rewards and one-stage resolution of uncertainty.

Second, Grant [18] models probabilistically sophisticated preferences in Savage’s setting without assuming the first-order stochastic dominance for risk attitudes. Grant’s primitives, motivations, and axioms are all distinct from mine. In particular, his model does not accommodate preference reversals, ambiguity aversion, partition dependence, or comparative ignorance.

Third, transitivity is relaxed in various axiomatic regret theories (Fishburn [13], Sugden [34], and others). Besides monotonicity, these models require weaker forms of transitivity (such as D-Transitivity in Diecidue and Somasundaram [9]), which do not hold in examples like (3) and (4). Conversely, some of my assumptions are incompatible with regret theories either.
Note that all of the above approaches employ some non-standard evaluations of the payoffs in $X$. By contrast, my model assumes regular risk attitudes that comply with expected utility. Intransitive and non-monotonic behaviors are attributed to partition-dependent beliefs or sets of beliefs that vary across comparisons. Another novelty is that monotonicity and transitivity are relaxed together, but each of these conditions has a clear interpretation in terms of the components of the BPEU and BPMEU representations.

Examples like (1)–(4) suggest that people should have some cognitive limitations in contingent reasoning—a mental process that requires thinking about all possible uncertain events without knowing which of them are true.\footnote{Esponda and Vespa \cite{12} use variations in contingent reasoning to generate many common anomalies, such as overbidding in auctions, naive voting, both Ellsberg and Allais Paradoxes. It is plausible that such variations can also generate some of the rankings (1) and (2).} Perhaps the most famous observation of this sort is that bidders are less likely to play the weakly dominant truthful strategy in the second-price private-value auction than in the English ascending-price counterpart (Kagel, Harstad, and Levin \cite{20}). To explain this finding, Li \cite{26} defines obviously dominant strategies (e.g. truthful bids in the English auctions) and provides further empirical evidence that such strategies are selected at higher rates than non-obviously dominant ones. Li’s approach is based on different primitives and focuses on incomplete obviously dominant relations, which satisfy both monotonicity and transitivity. In Section 4, I discuss how some of Li’s ideas can be combined with the BPMEU model.

The bivariate beliefs $p(\pi, \tau)$ can be naturally compared with Ahn and Ergin’s \cite{1} model of framing contingencies. Their partition dependent expected utility (PDEU) identifies subjective beliefs $p$ that depend on a ‘framing’ partition $\pi$ via Tversky and Koehler’s \cite{37} support theory: for any $\pi = \{E_1, \ldots, E_n\}$ and event $E_i \in \pi$,

$$p(E_i, \pi) = \frac{\nu(E_i)}{\sum_{j=1}^n \nu(E_j)}$$

for some non-negative support function $\nu$ such that $\sum_{E_i \in \pi} \nu(E_i) > 0$. The ‘framing’ partition $\pi$ in Ahn and Ergin’s analysis is imposed exogenously via some observable verbal announcements. Accordingly, PDEU has different primitives and represents preferences over act-partition pairs $(f, \pi)$ such that $\pi$ refines the partition $\pi f$ generated by $f$. The partition $\pi f$ generated by $f$. Moreover, PDEU assumes monotonicity and excludes ambiguity aversion. In Section 4.2, I adapt Ahn and Ergin’s main result (Theorem 3) to obtain the support theory (7) as a subcase of the BPEU model.

Besides comparative ignorance, agents may exhibit other forms of source dependence when their preferences comply with expected utility over $\pi$-measurable acts and over $\tau$-measurable acts, but become ambiguity averse when the two sources of uncertainty $\pi$ and $\tau$ are combined. The models of source dependence by Chew and Sagi \cite{7} and by Gul and Pesendorfer \cite{19} assume both transiti-
ity and monotonicity. However, preference reversals and non-monotonic choices should be plausible when the evaluation of a prospect \( f \) can depend on the sources of uncertainty involved in the alternative \( g \).

The BPMEU representation can be related to the literature on coarse and unforeseen contingencies (e.g. Mukerji [28], Epstein, Marinacci, and Seo [10]). A common theme in this literature is that the coarse perception of possible states and outcomes can motivate ambiguity averse behaviors and preference for hedging. The BPMEU model allows the decision maker to vary ambiguity aversion based on the coarseness—and other features—of the partitions \( \pi_f \) and \( \pi_g \).

Separable and resolute partitions can be compared with the various notions of unambiguous (crisp) events in Epstein and Zhang [11], Zhang [39], Ghirardato, Maccheroni, and Marinacci [15], Klibanoff et al. [22]. My definitions use transitivity and monotonicity principles to identify subdomains where beliefs and sets of beliefs are preserved across various contexts, while the unambiguous properties are identified via the Sure-Thing Principle and Independence conditions.

## 2 Preliminaries


Let \( \Omega = \{\omega, \ldots\} \) be a finite state space. Subsets \( E \subset \Omega \) are called events.

Let \( X = \{x, y, z, \ldots\} \) be a set of outcomes that may be obtained contingent on any state \( \omega \in \Omega \). Assume that \( X \) is a convex subset of some linear space. For example, \( X \) may be the space \( X_D \) of all lotteries—probability distributions with finite support on some exogenous set \( D \) of deterministic prizes. Note that \( X_D \) is a convex subset of the linear space \( \mathbb{R}^D \).

Let \( \mathcal{H} = \{f, g, h \ldots\} \) be the set of all acts—functions \( f : \Omega \to X \). Interpret each act \( f \in \mathcal{H} \) as a physical action that delivers outcomes \( f(\omega) \) contingent on the state \( \omega \in \Omega \). Each outcome \( x \in X \) is identified with the constant act \( x \in \mathcal{H} \).

Let \( \Delta = \{p, q, \ldots\} \) be the simplex of all probability distributions on \( \Omega \). For any \( f \in \mathcal{F} \) and \( q \in \Delta \), let

\[
f(q) = \sum_{\omega \in \Omega} q(\omega) f(\omega)
\]

be the mixture of the outcomes \( f(\omega) \) induced by the distribution \( q \). If \( X = X_D \) is a lottery space, then \( f(q) \) is the reduction of the compound lottery that delivers \( f(\omega) \) with probabilities \( q(\omega) \).

For any \( f, g \in \mathcal{H} \) and \( \alpha \in [0, 1] \), define a mixture \( \alpha f + (1 - \alpha) g \in \mathcal{H} \) as

\[
[\alpha f + (1 - \alpha) g](\omega) = \alpha f(\omega) + (1 - \alpha) g(\omega) \quad \text{for all } \omega \in \Omega.
\]

Note that \( [\alpha f + (1 - \alpha) g](q) = \alpha f(q) + (1 - \alpha) g(q) \) for all \( q \in \Delta \).

Let \( \mathcal{U} \) be the set of all non-constant functions \( u : X \to \mathbb{R} \) such that

\[
u(\alpha x + (1 - \alpha) y) = \alpha u(x) + (1 - \alpha) u(y)
\]
Call such functions \( u \in \mathcal{U} \) linear. If \( X = X_D \), then \( \mathcal{U} \) is the set of all non-constant vNM expected utility functions. A function \( v \in \mathcal{U} \) is a \textit{positive linear transformation} (plt) of \( u \in \mathcal{U} \) if \( v = \alpha u + \beta \) for some \( \alpha > 0 \) and \( \beta \in \mathbb{R} \).

Consider a decision maker (DM) who has a \textit{weak preference} \( \succeq \) over \( \mathcal{H} \) with asymmetric and symmetric parts \( > \) and \( \sim \) respectively. Assume throughout that
\begin{itemize}
  \item \( \succeq \) is \textit{complete}: for all \( f, g \in \mathcal{H} \), either \( f \succeq g \) or \( g \succeq f \),
  \item \( > \) is not empty: \( f > g \) for some \( f, g \in \mathcal{H} \).
\end{itemize}

Thus I do not attempt to model incomplete preferences in this paper.

Write \( f \succeq g \) if \( f(\omega) \succeq g(\omega) \) for all \( \omega \in \Omega \). Say that
\begin{itemize}
  \item \( \succeq \) is \textit{transitive} if for all \( f, g, h \in \mathcal{H} \), \( f \succeq g \succeq h \) implies \( f \succeq h \),
  \item \( \succeq \) is \textit{monotonic} if for all \( f, g \in \mathcal{H} \), \( f \succeq g \) implies \( f \succeq g \),
  \item \( \succeq \) is \textit{Archimedean} if for all \( f, g, h \in \mathcal{H} \) such that \( f > g \), there are \( \alpha, \beta \in (0, 1) \) such that \( \alpha f + (1 - \alpha)h \succ g \) and \( f \succ \beta g + (1 - \beta)h \),
  \item \( \succeq \) satisfies \textit{Independence} if for all \( f, g, h \in \mathcal{H} \) and \( \alpha \in (0, 1] \),
    \[ f \succeq g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h. \]
\end{itemize}

Anscombe-Aumann’s classic result implies that conditions (i)–(iv) are necessary and sufficient for \( \succeq \) to have a \textit{subjective expected utility} representation
\[ U(f) = u(f(p)), \]
where \( p \in \Delta \) is a unique subjective belief, and \( u \in \mathcal{U} \) is a linear utility index that is unique up to a plt.

All conditions (i)–(iv) are relaxed below.

**Axiom (Regularity).** \( \succeq \) \textit{has a linear representation} \( u \in \mathcal{U} \) over \( X \).

Regularity assumes transitivity, Archimedean, and Independence on the domain \( X \) of constant acts, but does not constrain any comparisons that involve non-constant prospects. If \( X = X_D \) is a lottery space, then Regularity asserts that the DM’s \textit{risk attitude} conforms with the vNM expected utility model.

Next, consider a weak form of monotonicity on the general domain \( \mathcal{H} \).

**Axiom (Range Monotonicity (RM)).** \textit{For all} \( f, g \in \mathcal{H} \) \textit{and} \( x, y \in X \),
\[ f \succeq x \succeq y \succeq g \implies f \succeq g \]
\[ f \succeq x > y \succeq g \implies f > g. \]
This condition is plausible even if the DM is unable or unwilling to formulate the state space $\Omega$ and to assign subjective beliefs to any non-trivial events $E \subset \Omega$. To comply with RM, she needs to determine only the best and worst possible payoffs of the acts $f$ and $g$ and then apply her state-invariant, complete, and transitive ranking of $X$. The rankings $f \succeq (\succ) g$ are compelling when the worst outcome of $f$ is weakly (strictly) better than the best outcome of $g$, or equivalently, $f \succeq x \succeq (\succ) y \succeq g$ for some $x, y \in X$.

RM can be related to Obvious Dominance (Li [26]) in mechanism design. Li’s condition identifies strategies that can be recognized as weakly dominant even under bounded rationality. Similarly, RM is still consistent with various cognitive limitations that would violate standard monotonicity. RM is equivalent to Obvious Monotonicity in Luyao and Levin [40] who use Anscombe-Aumann’s framework to obtain the transitive and certainty independent subcase of representation (9) below.

Say that $\succeq$ has certainty equivalents if for any $f \in H$, there is $x \in X$ such that $f \sim x$.

The following result provides a simple general benchmark for various representations that depart from the monotonicity and transitivity assumptions.

**Theorem 1.** $\succeq$ satisfies Regularity and RM if and only if there are functions $u \in U$ and $b : H \times H \to \Delta$ such that

$$f \succeq g \iff u(f(b(f,g))) \geq u(g(b(g,f)))$$

for all $f, g \in H$. Moreover,

(i) $u$ is unique up to a plt,

(ii) $\succeq$ is monotonic iff $\succeq$ is represented by (8) where $b(f,g) = b(g,f)$ for all $f, g \in H$,

(iii) $\succeq$ is transitive and has certainty equivalents iff $\succeq$ has a utility representation

$$U(f) = u(f(q(f)))$$

for some $u \in U$ and $q : H \to \Delta$.

All proofs are in the appendix.

Theorem 1 axiomatizes the bivariate expected utility (BEU) model (8) where

- outcomes in $X$ are ranked via a vNM utility index $u \in U$,
- any two acts $f$ and $g$ are compared via bivariate beliefs $b(f,g)$ and $b(g,f)$ respectively,
- each belief $b(f,g) \in \Delta$ depends on two variables—the evaluated act $f$ and the alternative $g$—and need not equal $b(g,f)$,
preferences need not be transitive, monotonic, continuous aside from the domain $X$ of constant acts.

Moreover, Theorem 1 interprets monotonicity and transitivity in terms of the belief function $b : \mathcal{H} \times \mathcal{H} \rightarrow \Delta$. Monotonicity implies that $b$ should be symmetric so that $b(f, g) = b(g, f)$ for all $f, g \in \mathcal{H}$. Assuming certainty equivalents, transitivity implies that the belief function $b(f, g) = q(f)$ is univariate and unchanged by $g$. In this case, the BEU model (8) becomes a utility representation (9).

For example, the multiple priors model of Gilboa and Schmeidler [17] is a special case of (9) where $q(f)$ is selected as the least favorable element in a subjective set $M \subset \Delta$,

$$U(f) = \min_{q \in M} u(q).$$

This model is both transitive and monotonic.

3 Main Representation Results

Any pair of bivariate beliefs $b(f, g)$ and $b(g, f)$ in the BEU model (8) is restricted only by one observation, such as $f \succ g$. However, this lone comparison is insufficient to identify $b(f, g)$ or $b(g, f)$ uniquely on any non-trivial event other than $\Omega$ and $\emptyset$. Thus the generality of the BEU model can be an embarrassment of riches in applications.

Next, I characterize several refinements of the BEU model that do not assume transitivity or monotonicity, but identify beliefs $b(f, g)$ and $b(g, f)$ uniquely on a suitable class of events for each generic pair of acts $f$ and $g$. These refinements require additional notation and terminology.

Let $\Pi = \{\pi, \tau, \sigma \ldots \}$ be the set of all partitions of the state space $\Omega$. Each partition $\pi \in \Pi$ is a collection $\{E_1, \ldots, E_k\}$ of non-empty disjoint events such that $\Omega = E_1 \cup E_2 \cup \cdots \cup E_k$.

For any act $f \in \mathcal{F}$, outcomes $x \in X$, and partitions $\pi, \tau \in \Pi$,

- $f^{-1}(x) = \{\omega \in \Omega : f(\omega) = x\}$,
- the partition $\pi(f) \in \Pi$, or $\pi f$ for short, is generated by $f$ if it consists of all non-empty events $E$ such that $E = f^{-1}(y)$ for some $y \in X$,
- $\mathcal{A}_{\tau}$ is the set of all acts $f \in \mathcal{H}$ such that $\pi f = \tau$,
- $f$ is analogous to $f' \in \mathcal{H}$ if $\pi f' = \pi f$,
- $\mathcal{A}_f = A_{\pi f}$ is the set of all acts $f' \in \mathcal{H}$ that are analogous to $f$,
- the cross-partition $\pi \vee \tau \in \Pi$ consists of all non-empty events $E = E \cap T$ such that $E \in \pi$ and $T \in \tau$,
• if $\pi = \pi \lor \tau$, say that $\pi$ is finer than (refines) $\tau$, and $\tau$ is coarser than $\pi$.

Partition dependence and other behavioral effects suggest that the comparison between any two prospects $f, g \in H$ can be affected by the events and partitions that are used to specify $f$ and $g$.

Assume that each act $f \in H$ is described as a function $f_\pi : \pi f \to X$ such that

$$f(s) = f_\pi(E) \text{ for all } E \in \pi f \text{ and } s \in E. \quad (10)$$

This description is minimal in the sense that it does not require any additional details about states or events beyond the partition $\pi_f$. For example, a binary bet

$$xEy = \begin{cases} x & \text{if } \omega \in E \\ y & \text{if } \omega \notin E \end{cases}$$

is naturally specified by the combination of an event $E \subset \Omega$, its complement $E^c = \Omega \setminus E$, and two payoffs $x$ and $y$ that are received contingent on these two events. Listing all states within $E$ and $E^c$ is not necessary. In financial settings, investors can contemplate betting on events like “the stock market will crash by more than 10% next month” or “the interest rate will be raised at the next Fed meeting” without listing all conceivable realizations of stock prices or Fed announcements that belong to these events.

By definition, the descriptions (10) of any analogous acts $f \in H$ and $f' \in A_f$ employ the same partition $\pi_f = \pi_{f'}$. This similarity motivates calling $f$ and $f'$ analogous, even though $f$ and $f'$ can assign arbitrary distinct outcomes to events in the generated partition $\pi_f$.

In some empirical examples, comparisons between prospects $f$ and $g$ can be framed by explicit announcements that do not change the partitions $\pi_f$ or $\pi_g$ and hence, cannot be captured by descriptions (10). Similarly to Ahn and Ergin [1], one can take such announcements as an extra primitive of the model. My representation results can be adapted for such settings as well (see Section 4.2 for further discussion.)

### 3.1 Bi-partitional Maxmin Expected Utility (BPMEU)

Relax the assumptions of the multiple priors model as follows.

**Axiom 1 (Partitional Transitivity (PT)).** For all $f, g \in H$, $f' \in A_f$, and $g' \in A_g$,

$$f \succeq g' \succeq f' \succeq g \implies f \succeq g.$$

PT relaxes transitivity and allows cycles

$$f \succeq g' \succeq f' \succeq g \succ f \quad \text{where } f, f', g, g' \in H. \quad (11)$$
Here \( f \) is revealed as weakly better than \( f' \) in comparisons with \( g' \), but \( f' \) is revealed as strictly better than \( f \) in comparisons with \( g \).

On the other hand, PT prohibits all cycles (11) where \( \pi f = \pi f' \) and \( \pi g = \pi g' \). In this case, the two alternatives in each of the four comparisons \( f \succeq g' \succeq f' \succeq g \) and \( f \succeq g \) generate the same pair of partitions \( \pi f \) and \( \pi g \). Thus PT accommodates any observations with distinct pairs of generated partitions.

**Axiom 2** (Partitional Monotonicity (PM)). For all \( f, g, f', g' \in \mathcal{H} \) such that \( (f', g') \in X \times X \) or \( (f', g') \in \mathcal{A}_f \times \mathcal{A}_g \),

\[
\begin{align*}
    f \succeq f' \succeq g' \succeq g & \implies f \succeq g \\
    f \succeq f' \succ g' \succeq g & \implies f \succ g.
\end{align*}
\]

If \( (f', g') \in X \times X \) is a pair of constant acts, then PM is identical to RM.

To interpret the other cases, take any analogous acts \( f \in \mathcal{H} \) and \( f' \in \mathcal{A}_f \) such that \( f \succeq f' \). PM requires that neither of the rankings \( f' \succeq f \succeq g \) and \( f' \succeq g' \succeq f \) can hold for any \( g \in \mathcal{H} \). Indeed, \( g \succeq f \) implies that \( g \succeq g \succeq f \succeq f' \) and hence, \( g \succeq f' \) by PM. Similarly, \( f' \succeq g \) implies \( f \succeq f' \succeq g \succeq g \) and hence, \( f \succeq g \). Thus, \( f' \) cannot be revealed to be strictly better than an analogous act \( f \succeq f' \) via comparisons with any \( g \in \mathcal{H} \). This claim agrees with any observations with distinct pairs of generated partitions.

By PT and PM, the restriction of \( \succeq \) to any domain \( \mathcal{A}_f \) should be both transitive and monotonic.\(^2\) However, these axioms impose some additional discipline on comparisons between acts \( f \) and \( g \) when \( \pi f \neq \pi g \). This discipline is indispensable in the representation results below.

**Axiom 3** (Certainty Independence (CI)). For all \( \alpha \in (0, 1) \), \( f, g \in \mathcal{H} \), and \( x \in X \),

\[
f \succeq g \iff \alpha f + (1 - \alpha)x \succeq \alpha g + (1 - \alpha)x.
\]

CI is taken as is. This weak form of Independence is plausible for ambiguity aversion, but also for partition dependent behaviors. Note that the mixtures \( \alpha f + (1 - \alpha)x \) and \( \alpha g + (1 - \alpha)x \) are analogous to \( f \) and \( g \) respectively. Thus CI cannot be refuted by choices with distinct pairs of generated partitions.

**Axiom 4** (Certainty Archimedean (CA)). For any \( f, g \in \mathcal{H} \) and \( x \in X \), if \( f \succ g \), then \( f \succ \alpha g + (1 - \alpha)x \) and \( \beta f + (1 - \beta)x \succ g \) for some \( \alpha, \beta \in (0, 1) \).

Unlike the standard Archimedean condition, CA is restricted to mixtures \( \alpha g + (1 - \alpha)x \) with constant acts \( x \in X \). The acts \( g \) and \( \alpha g + (1 - \alpha)x \) are analogous for all \( \alpha \in (0, 1) \). By contrast, if \( h \in \mathcal{H} \) is not constant, then \( g \) and the mixture \( \alpha g + (1 - \alpha)h \) need not be analogous for \( \alpha \in (0, 1) \). Thus partition dependence

\(^2\)For any \( f', g \in \mathcal{A}_f \), \( f \succeq f' \succeq g \succeq g \) implies \( f \succeq g \), and \( f \succeq f' \succeq f' \succeq f' \) implies \( f \succeq f' \).
can motivate discontinuous preferences \( a g + (1 - a) h \succeq f \succ g \) for all \( a \in (0, 1) \), which is allowed by CA.\(^3\)

Another familiar principle, Gilboa–Schmeidler’s Uncertainty Aversion, is relaxed as follows.

**Axiom 5 (Partitional Uncertainty Aversion (PUA)).** For all \( a \in [0, 1] \) and \( f, g, h \in \mathcal{H} \) such that \( g \in \mathcal{A}_f \) and \( a f + (1 - a) g \in \mathcal{A}_f \),

\[
 f \succeq h \quad \text{and} \quad g \succeq h \quad \Rightarrow \quad a f + (1 - a) g \succeq h.
\]

PUA imposes Uncertainty Aversion when the analogous acts \( f, g, \alpha f + (1 - \alpha) g \) are ranked against an arbitrary alternative \( h \). Given that \( f \succeq h \) and \( g \succeq h \), the ranking \( \alpha f + (1 - \alpha) g \succeq h \) is motivated by ambiguity aversion and is also consistent with any observations where pairs of generated partitions are distinct.

Axioms 1–5 can be derived from assumptions on the bivariate beliefs \( b(\cdot) \) in the general BEU model. Suppose that for all acts \( f, f', g, g' \in \mathcal{H} \),

B1: if \( \pi g = \pi g' \), then \( b(f, g) = b(f, g') \),

B2: if \( \pi f = \pi f' \), then \( f(b(f, g)) \preceq f(b(f', g)) \).

Both conditions are imposed on beliefs \( b(f, g) \), \( b(f, g') \), and \( b(f', g) \) where \( f \) is analogous with \( f' \), and \( g \) is analogous with \( g' \). In this way, B1 and B2 avoid partition dependent effects on beliefs. B1 assumes that \( f \) should be evaluated via the same belief \( b(f, g) = b(f, g') \) when compared with alternatives \( g \) and \( g' \) that are analogous. B2 assumes that analogous acts \( f \) and \( f' \) can be evaluated via distinct beliefs \( b(f, g) \) and \( b(f', g) \) when compared to \( g \), but \( b(f, g') \) should not be more favorable for \( f \) than \( b(f', g) \) because of ambiguity aversion.

Axioms 1–5 follow from B1 and B2, which I show in Section 4.5. For example, PT is implied by B1 alone. Take any acts \( f, g \in \mathcal{H} \), \( f' \in \mathcal{A}_f \) and \( g' \in \mathcal{A}_g \) such that \( f \succeq g' \succeq f' \succeq g \). By B1,

\[
\begin{align*}
 f(b(f, g)) &= f(b(f, g')) \succeq g'(b(g', f)) = g'(b(g', f')) \\
 f'(b(f', g')) &= f'(b(f', g)) \succeq g(b(g, f')) = g(b(g, f))
\end{align*}
\]

and hence, \( f \succeq g \).

For any probability distribution \( q \in \Delta \) and partition \( \pi \in \Pi \), let \( q_\pi \) be the projection of \( q \) to \( \pi \) such that

\[
 q_\pi(E) = q(E) \quad \text{for all} \ E \in \pi.
\]

Let \( \Delta_\pi \) be the set of all projections \( q_\pi \) for \( q \in \Delta \).

Let \( \mathcal{M} = \{M, \ldots\} \) be the set of all non-empty, closed, and convex subsets \( M \subset \Delta \). For any \( M \in \mathcal{M} \), let \( M_\pi \) be the set of all projections \( q_\pi \) of \( q \in M \).

Axioms 1–5 extend the multiple priors model as follows.

\(^3\)Like any continuity condition, CA is consistent with any finite number of observations. Thus it can be empirically refuted only in combination with other axioms.
Theorem 2. $\succeq$ satisfies Axioms 1–5 if and only if there are functions $u \in U$ and $M : \Pi \times \Pi \to \mathcal{M}$ such that

$$f \succeq g \iff \min_{q \in M(\pi f, \pi g)} u(f(q)) \geq \min_{q \in M(\pi g, \pi f)} u(g(q))$$

for all acts $f, g \in H$. Moreover,

(i) $u$ is unique up to a plt, and the set $M_\pi(\pi, \tau) \subset \Delta_\pi$ is unique for all $\pi, \tau \in \Pi$,

(ii) $\succeq$ is monotonic iff $\succeq$ is represented by (13) where $M(\pi, \tau) = M(\tau, \pi)$ for all $\pi, \tau \in \Pi$,

(iii) $\succeq$ is transitive iff $\succeq$ is represented by (13) where $M(\pi, \tau) = M(\pi', \tau')$ for all $\pi, \tau, \pi', \tau' \in \Pi$,

(iv) $\succeq$ is monotonic and transitive iff $\succeq$ is represented by (13) where $M(\pi, \tau) = M(\pi', \tau')$ for all $\pi, \tau, \pi', \tau' \in \Pi$.

The DM as portrayed by (13) compares any pair of acts $f, g \in H$ via the least favorable probabilistic scenarios selected from the sets $M(\pi f, \pi g)$ and $M(\pi g, \pi f)$ respectively. These sets are determined by the partitions $\pi = \pi f$ and $\tau = \pi g$ generated by the feasible prospects $f$ and $g$. Accordingly, representation (13) is called bi-partitional maxmin expected utility (BPMEU).

The uniqueness claim in Theorem 2 asserts that $\succeq$ has two distinct representations (13) with components $(u, M)$ and $(u^*, M^*)$ where $u, u^* \in U$ and $M, M^* : \Pi \times \Pi \to \mathcal{M}$ if and only if $u^*$ is a plt of $u$, and $M^*_\pi(\pi, \tau) = M_\pi(\pi, \tau)$ for all $\pi, \tau \in \Pi$. Therefore, the preference $\succeq$ identifies each projection $M_\pi(\pi, \tau)$ uniquely on the corresponding partition $\pi$. However, if the pairs $(\pi, \tau)$ and $(\pi', \tau')$ are distinct, then the projection $M_\pi(\pi', \tau')$ and $M_\pi(\pi, \tau)$ need not be equal or related in any other way.

Transitivity and monotonicity are interpretable in terms of the function $M$. Monotonicity requires that any two partitions $\pi$ and $\tau$ should have symmetric effects on the sets of beliefs $M(\pi, \tau) = M(\tau, \pi)$.

In the transitive case, $\succeq$ must have a utility representation

$$U(f) = \min_{q \in Q(\pi f)} u(f(q))$$

where $Q : \Pi \to \Delta$ is a univariate function such that $Q(\pi) = M(\pi, \pi)$ for all $\pi \in \Pi$. The combination of monotonicity and transitivity turns BPMEU into the standard multiple priors model.

### 3.2 Bi-partitional Expected Utility (BPEU)

The BPMEU model can be refined further in various directions. First, ambiguity aversion can be excluded via the following condition.
Axiom 6 (Partitional Betweenness (PB)). For all \( \alpha \in [0, 1] \) and \( f, g, h, h' \in \mathcal{H} \) such that \( g \in \mathcal{A}_f \) and \( \alpha f + (1 - \alpha)g \in \mathcal{A}_f \),

\[
h' \succeq f \succeq h \quad \text{and} \quad h' \succeq g \succeq h \implies h' \succeq \alpha f + (1 - \alpha)g \succeq h.
\]

Similarly to PUA, PB is imposed on comparisons of analogous acts \( f, g, \alpha f + (1 - \alpha)g \) with arbitrary alternatives \( h \) and \( h' \). Given the rankings \( h' \succeq f \succeq h \) and \( h' \succeq g \succeq h \), the preference \( h' \succeq \alpha f + (1 - \alpha)g \succeq h \) is motivated by the standard separability argument and is consistent with any observations with distinct pairs of generated partitions.

Note that in Anscombe–Aumann’s expected utility model, the Independence axiom can be replaced by the combination of CI and a version of Dekel’s [8] Betweenness: for all \( \alpha \in (0, 1) \) and \( f, g \in \mathcal{H} \),

\[
f \succeq g \implies f \succeq \alpha f + (1 - \alpha)g \succeq g.
\]

In the partition dependent case, Independence is replaced by CI and PB.

To interpret PB in terms of the bivariate beliefs in the BEU model, assume

B3: \( b(f, g) = b(f', g') \) for all acts \( f, f', g, g' \in \mathcal{H} \) such that \( \pi f = \pi f' \) and \( \pi g = \pi g' \).

This assumption strengthens both B1 and B2 and hence, implies Axioms 1–5. It implies PB as well (see Section 4.5).

Theorem 3. \( \succeq \) satisfies Axioms 1-4 and PB if and only if there are functions \( u \in \mathcal{U} \) and \( p : \Pi \times \Pi \to \Delta \) such that

\[
f \succeq g \iff u(f(p(\pi f, \pi g))) \geq u(g(p(\pi g, \pi f)))
\]

for all acts \( f, g \in \mathcal{H} \). Moreover,

(i) \( u \) is unique up to a plt, and the projections \( p_\pi(\pi, \tau) \in \Delta_\pi \) are unique for all \( \pi, \tau \in \Pi \),

(ii) \( \succeq \) is monotonic iff \( \succeq \) is represented by (14) where \( p(\pi, \tau) = p(\tau, \pi) \) for all \( \pi, \tau \in \Pi \),

(iii) \( \succeq \) is transitive iff \( \succeq \) is represented by (14) where \( p(\pi, \tau) = p(\pi, \pi) \) for all \( \pi, \tau \in \Pi \),

(iv) \( \succeq \) is monotonic and transitive iff \( \succeq \) is represented by (14) where \( p(\pi, \tau) = p(\pi', \tau') \) for all \( \pi, \tau, \pi', \tau' \in \Pi \).

Theorem 3 characterizes a special case of the BEU model where the beliefs \( b(f, g) = p(\pi f, \pi g) \) are determined exclusively by the pair of generated partitions \( \pi = \pi f \) and \( \tau = \pi g \). Accordingly, (14) is called a bi-partitional expected utility (BPEU) representation.
Unlike Theorem 1, the BPEU model identifies the projection of any bivariate belief \( p(\pi, \tau) \) to the corresponding partition \( \pi \) uniquely: \( \succeq \) can have two distinct representations (14) with components \((u, p)\) and \((u', p')\) where \( u, u' \in \mathcal{U} \) and \( p, p' : \Pi \times \Pi \to \Delta \) if and only if \( u' \) is a plt of \( u \), and \( p'_\pi(\pi, \tau) = p_\pi(\pi, \tau) \) for all \( \pi, \tau \in \Pi \). However, the preference \( \succeq \) does not dictate how the projection \( p_\pi(\pi, \tau) \) should be extended from \( \pi \) to a probability measure on the entire state space \( \Omega \). In fact, \( \succeq \) is consistent with any such extension in its BPEU representation. Moreover, if the pairs \((\pi, \tau)\) and \((\pi', \tau')\) are distinct, then the projections \( p_\pi(\pi, \tau) \) and \( p_\pi(\pi', \tau') \) need not be equal or related in any way.\(^4\)

Similarly to Theorem 1, transitivity and monotonicity provide additional constraints for the belief function \( p : \Pi \times \Pi \to \Delta \). Monotonicity guarantees the symmetry of the belief function \( p(\pi, \tau) = p(\tau, \pi) \) for all partitions \( \pi \) and \( \tau \). Transitivity asserts that \( \succeq \) has a utility representation
\[
U(f) = u(f(q(\pi f)))
\]
where \( q : \Pi \to \Delta \) is a univariate belief function such that \( q(\pi) = p(\pi, \pi) \) for all \( \pi \in \Pi \). The combination of monotonicity and transitivity turns BPEU into the standard expected utility model.

### 3.3 Cross-Partitional Models

It can be analytically convenient to model partition dependence in terms of the cross-partition \( \pi f \lor \pi g \) rather than the pair of generated partitions \( \pi_f \) and \( \pi_g \). For instance, the cross-partitional structure provides natural extensions for choices among more than two alternatives (see Section 4.4 below).

By definition, \( \pi_f \lor \pi_g \) consists of all non-empty events that have the form \( f^{-1}(x) \cap f^{-1}(y) \) for some \( x \) and \( y \). These events form the coarsest partition \( \pi \) that allows to identify both acts \( f \) and \( g \) with some functions \( f_s, g_s : \pi \to X \).

To obtain cross-partitional representations, strengthen PT as follows.

**Axiom 7 (Cross-Partitional Transitivity (CPT)).** For all \( f, g, f', g' \in \mathcal{H} \) such that
\[
\pi f \lor \pi g = \pi f' \lor \pi g = \pi f \lor \pi g' = \pi f' \lor \pi g',
\]
\( f \succeq g' \succeq f' \succeq g \) implies \( f \succeq g \).

CPT prohibits cycles \( f \succeq g' \succeq f' \succeq g \succ f \) whenever the generated cross-partitions are the same for each of the four comparisons in this cycle.

CPT can be derived from another condition on the bivariate beliefs in the BEU model (8). Assume that for all acts \( f, g, g' \in \mathcal{H} \),

\(^4\)The total numbers of partitions of \( n \)-element sets are given by the Bell numbers \( B_n \), so that \( B_2 = 2 \), \( B_3 = 5 \), \( B_4 = 15 \), \( B_5 = 52 \), \( B_6 = 203 \), etc. Given any state \( \omega \in \Omega \), there are \( B_{\Omega - 1} \) partitions such that \( \{ \omega \} \in \pi \) and hence, \( B_\Omega B_{\Omega - 1} \) distinct pairs \((\pi, \tau)\) for which \( \{ \omega \} \in \pi \).

Thus the probability \( p(\pi, \tau)(\omega) \) can take up to \( B_\Omega B_{\Omega - 1} \) distinct values for various \( \pi \) and \( \tau \). The number \( B_\Omega B_{\Omega - 1} = O(\Omega^2 e^\Omega) \) grows exponentially with the size of the state space \( \Omega \).
B1*: if $\pi f \lor \pi g = \pi f \lor \pi g'$, then $b(f, g) = b(f, g')$.

CPT follows from B1*. Assume (15) and $f \succeq g' \succeq f' \succeq g$. B1* implies (12) and hence, $f \succeq g$.

**Theorem 4.** $\succeq$ satisfies CPT and Axioms 2–5 iff $\succeq$ is represented by (13) where

$$M(\pi, \tau) = M(\pi, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi. \quad (16)$$

Moreover,

(i) $\succeq$ is monotonic iff $\succeq$ has a BPMEU representation (13) where

$$M(\pi, \tau) = M(\pi \lor \tau, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi, \quad (17)$$

(ii) $\succeq$ satisfies PC iff $\succeq$ has a BPEU representation (14) where

$$p(\pi, \tau) = p(\pi, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi.$$

(iii) $\succeq$ is monotonic and satisfies PC iff $\succeq$ has a BPEU representation (14) where

$$p(\pi, \tau) = p(\pi \lor \tau, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi.$$

Theorem 4 identifies special cases of BPMEU and BPEU representations where the beliefs $p(\pi, \tau)$ and sets of beliefs $M(\pi, \tau)$ are unchanged when $\tau$ is replaced by $\pi \lor \tau$. The structures (16) and (17) are called cross-partitional maxmin expected utility (CPMEU) and cross-partitional expected utility (CPEU) respectively.

In the monotonic case, the bivariate functions $M$ and $p$ can be replaced by univariate mappings $Q : \Pi \to \mathcal{M}$ and $q : \Pi \to \Delta$ respectively such that for all $\pi, \tau \in \Pi$, $M(\pi, \tau) = Q(\pi \lor \tau)$ and $p(\pi, \tau) = q(\pi \lor \tau)$.

### 3.4 Resolute and Separable Partitions

It is natural to ask when all the revealed subjective beliefs $p(\pi, \tau)$ or sets of beliefs $M(\pi, \tau)$ in the BPEU or BPMEU models should agree on a given partition $\theta \in \Pi$. Say that an act $h \in \mathcal{H}$ is $\theta$-measurable if $\pi h$ is coarser than $\theta$.

For any $\theta \in \Pi$, let $\mathcal{H}_{\theta}$ be the set of all $\theta$-measurable acts $h \in \mathcal{H}$. Note that the equality $\pi h = \theta$ is sufficient, but not necessary for $h$ to be $\theta$-measurable. In other words, $\mathcal{A}_\theta$ is a proper subset of $\mathcal{H}_\theta$.

Call a partition $\theta \in \Pi$ resolute if for all $f, g \in \mathcal{H}$ and $h, h' \in \mathcal{H}_\theta$,

$$f \succeq h \succeq h' \succeq g \quad \Rightarrow \quad f \succeq g. \quad (18)$$

Otherwise, say that $\theta$ is irresolute.
Call a partition \( \theta \in \Pi \) separable if \( \theta \) is resolute, and \( \succeq \) satisfies Independence over \( \theta \)-measurable acts: for all \( f, g, h \in \mathcal{H}_\theta \) and \( \alpha \in (0,1] \),

\[
f \succeq g \iff \alpha f + (1- \alpha) h \succeq \alpha g + (1- \alpha) h.
\]

Otherwise, \( \theta \) is called inseparable. By definition, if a partition \( \pi \in \Pi \) is coarser than some resolute (separable) partition \( \theta \), then \( \pi \) is resolute (separable) because \( \mathcal{H}_\pi \subset \mathcal{H}_\theta \).

Let \( \Pi_r \subset \Pi \) be the family of all resolute partitions. Let \( \Pi_s \subset \Pi_r \) be the subfamily of all separable partitions. The general BEU model implies that the trivial partition \( \theta = \{ \Omega \} \) is separable and a fortiori, resolute. Thus \( \Pi_s \) and \( \Pi_r \) are not empty.

**Theorem 5.** If \( \succeq \) satisfies Axioms 1–5, then there is a set \( Q^* \in \mathcal{M} \), a probability measure \( q^* \in Q \), and functions \( u \in U \), \( M: \Pi \times \Pi \to \mathcal{M} \) such that

(i) \( \succeq \) is represented by the BPMEU model (13),

(ii) \( M_{q^*}(\pi, \tau) = Q^* \) for all \( \pi, \tau \in \Pi \) and resolute partitions \( \theta \in \Pi_r \),

(iii) \( M_{q^*}(\pi, \tau) = \{ q^*_\theta \} \) for all \( \pi, \tau \in \Pi \) and separable partitions \( \theta \in \Pi_s \subset \Pi_r \).

Moreover, if \( \succeq \) satisfies Axioms 1–4 and PB, then all resolute partitions are separable, and there is \( p: \Pi \times \Pi \to \Delta \) such that

(iv) \( \succeq \) is represented by the BPEU model (14),

(v) \( p_\theta(\pi, \tau) = q^*_\theta \) for all \( \pi, \tau \in \Pi \) and separable partitions \( \theta \in \Pi_s = \Pi_r \).

This theorem motivates the endogenous definitions of resolute and separable partitions and uses these notions to obtain additional structure for the functions \( M(\cdot) \) and \( p(\cdot) \) in the BPMEU and BPEU representations.

Fix a partition \( \theta \in \Pi \) and suppose that all sets of priors \( M(\pi, \tau) \) in the BPMEU model share the same projection \( Q^*_\theta \subset \Delta_\theta \) to \( \theta \). Then \( \theta \) must be resolute. Indeed, if \( f, g, h \in \mathcal{H} \) and \( h, h' \in \mathcal{H}_\theta \) satisfy \( f \succeq h \succeq h' \succeq g \), then

\[
\min_{q \in M(\pi f, \pi g)} u(f(q)) \geq \min_{q \in M(\pi f, \pi g)} u(h(q)) = \min_{q \in M(\pi f, \pi g)} u(h(q_\theta)) = \min_{q \in M(\pi h, \pi h')} u(h(q)) \geq \min_{q \in M(\pi h, \pi h')} u(h(q_\theta))
\]

and hence, \( f \succeq g \). Moreover, the restriction of \( \succeq \) to \( \theta \)-measurable acts \( h \in \mathcal{H}_\theta \) is represented by MEU

\[
U_\theta(h) = \min_{q \in Q^*_\theta} u(h(q_\theta)). \quad (19)
\]

If \( Q^*_\theta \) is a singleton, then \( \succeq \) should satisfy Independence over \( \mathcal{H}_\theta \), and hence \( \theta \) must be separable.

Theorem 5 asserts the opposite claims, which are harder to derive. In the BPMEU model, the function \( M: \Pi \times \Pi \to \mathcal{M} \) can be always selected so that for all \( \pi, \tau \in \Pi \),
• the projections $M_{\theta}(\pi, \tau)$ to all resolute partitions $\theta \in \Pi_r$ agree with an invariant set of priors $Q^* \in \mathcal{M}$,

• the projections $M_{\theta}(\pi, \tau)$ to all separable partitions $\theta \in \Pi_s \subset \Pi_r$ agree with an invariant probability measure $q^* \in Q$.

Let $\mathcal{H}_r \subset \mathcal{H}$ and $\mathcal{H}_s \subset \mathcal{H}$ be the collections of all acts $h$ such that $\pi h \in \Pi_r$ is resolute or respectively, $\pi h \in \Pi_s$ is separable. Another corollary of Theorem 5 is that the utility representations (19) can be all combined on the domain $\mathcal{H}_r$ via

$$U(h) = \min_{q \in Q^*} u(h(q)).$$

When restricted to $\mathcal{H}_s$, the utility function $U$ takes the expected utility form $U(h) = u(h(q^*))$. In these extensions of the expected utility and multiple priors models, the domains $\mathcal{H}_r$ and $\mathcal{H}_s$ need not be mixture spaces, and preferences over the entire $\mathcal{H}$ can violate all of Gilboa-Schmeidler’s axioms, except for completeness and CI.

4 Discussion

Various behavioral phenomena fit the BPMEU model and its refinements.

To illustrate, consider some examples from the introduction. Fix any two rewards $x \succ y$. The BPEU model allows any rankings of binary bets $x E y$ because one can always find values $p(E, E')$, $p(E', E) \in [0, 1]$ such that

$$x E y \succeq x E' y \Rightarrow p(E, E') \geq p(E', E).$$

In particular, non-monotonic choices like (1) are allowed. To put some constraints on the weights $p(E, E')$, one may vary the outcomes $x$ and $y$, or impose some separable partitions. For example, let $\Omega = \{G_i, R_i\}_{i=1}^k$ be the set of all states that are determined by $k$ drawings from a two-colored urn. Then non-monotonic choices like (1) can be revealed together with the known probabilities $p(G) = \frac{2}{3}$ and $p(R) = \frac{1}{3}$ in every single coin flip. Assume that all partitions $\theta_i = \{G_i, R_i\}$ that correspond to a single coin flip are separable, and $10G_1, 0 \sim 20R_0$ where the payoffs are in utils. Then $p(G_i, \pi, \tau) = \frac{2}{3}$ and $p(R_i, \pi, \tau) = \frac{1}{3}$ for all coin flips $i$ and all partitions $\pi, \tau$. However, the inequality $p(E, \pi, \tau) > p(A, \tau, \pi)$ can hold for $\pi = \{E, E^c\}$, $\tau = \{A, A^c\}$, and events $A = \{RGRRR\} \supset \{RGRRRG\} = E$.

Recall example (2) where the non-monotonic ranking $f \succ g$ is motivated by the prevalence of large payoffs in the range of $f(\Omega)$ and low payoffs in $g(\Omega)$. For each partition $\pi = \{E_1, \ldots, E_n\}$, let $\sigma_{\pi} \in \Delta$ be the uniform salience probability measure such that $\sigma_{\pi}(E_i) = \frac{1}{n}$. Suppose that the beliefs $p(\pi, \tau)$ have the univariate parametric form

$$p(\pi, \tau) = p(\pi, \pi) = \lambda_{\pi} \sigma_{\pi} + (1 - \lambda_{\pi})p^*$$

(20)
where $p^* \in \Delta$ is a baseline probability measure, and $\lambda_\pi \in [0, 1]$ reflects the direct appeal (salience) of possible payoffs obtained on the partition $\pi$. Sonnemann et al. [33] fit their empirical data with a special case of (20) where $\lambda_\pi$ is invariant of $\pi$.

In the example (2), the ranking $f > g$ is implied by BPEU with the structure (20) where $p^*(\omega) = 0.01$ for all $\omega \in \Omega = \{1, \ldots, 100\}$ and $\lambda_\pi = 0.2$ for all partitions $\pi$.

4.1 Obvious Dominance

Li [26] provides empirical evidence that agents should find contingent reasoning easier when they focus on some special partitions $\pi$, which in Li’s papers are dictated by a mechanism structure. For preferences over state contingent acts, his definition of obvious dominance can be restated as follows.

Say that a partition $\theta$ is obvious if for all acts $f, g \in \mathcal{H}$ and $\theta$-measurable $h \in \mathcal{H}_\theta$, $f \succeq h \succeq g \Rightarrow f \succeq g$. (21)

Of course, obvious partitions include all resolute and separable ones.

The definition (21) captures Li’s intuition: if the worst outcome of the act $f$ is weakly better than the best outcome of the act $g$ contingent on every event in the special partition $\theta$, then $f$ should be chosen over $g$.

Yet the definition of obvious partitions does not restrain subjective beliefs $p(\pi, \tau)$ for some partitions $\pi$ and $\tau$. Thus it does not guarantee that these beliefs must be always preserved on $\theta$.

For example, suppose that the state $\Omega = \{G, R\} \times \{H, T\}$ is determined by a random drawing from a two-color urn and a coin flip. Suppose that $\theta = \{H, T\}$ is obvious, and let $\tau = \{G, R\}$. Then the beliefs $p(\theta, \tau)$ are arbitrary on $\theta$. Indeed for all $f, h \in \mathcal{H}_\theta$ and $g \in \mathcal{H}_\tau$, the dominance $f \succeq h \succeq g$ that there is a constant $x \in X$ such that $h \succeq x \succeq g$. Then $f \succeq g$ by RM or a fortiori, PM. Similarly, $g \succeq h \succeq f$ implies $g \succeq f$. Thus $\theta$ can be obvious for any beliefs $p(\theta, \tau)$ and $p(\tau, \theta)$.

Thus the stronger definitions of resolute and separable partitions appear more convenient in the decision theoretic framework.

4.2 Framing and Tversky-Koehler’s Support Theory

Recall the monotonic CPEU model

$$f \succeq g \iff u(f(q(\pi f \lor \pi g))) \geq u(g(q(\pi f \lor \pi g)))$$  (22)

for some functions $u \in \mathcal{U}$ and $q : \Pi \to \Delta$.

This representation is tightly related to the partition-dependent expected utility model (PDEU) proposed by Ahn and Ergin [1]. In their setting, preferences $\succeq^*$ are defined over pairs $(f, \pi) \in \mathcal{H} \times \Pi$ such that $\pi$ refines $\pi f$. By contrast, preferences $\succeq$ in my representations results are defined over the domain $\mathcal{H}$. 
Given a ranking $\succeq^*$ over pairs $(f, \pi)$, derive a binary relation $\succeq$ on $\mathcal{H}$ as

$$f \succeq g \iff (f, \pi f \lor \pi g) \succeq^* (g, \pi f \lor \pi g).$$

In my setting, $\succeq$ is observable, and $\succeq^*$ can be derived via representation (22) as

$$(f, \pi) \succeq^* (g, \tau) \iff u(f(q \lor \tau))) \geq u(g(q \lor \tau))).$$  

(Ahn and Ergin characterize representation (22) where the belief function $q$ satisfies Tversky and Koehler’s [37] support theory: for any partition $\pi = \{E_1, \ldots, E_n\}$ and event $E_i \in \pi$,

$$q(E_i, \pi) = \frac{\nu(E_i)}{\sum_{j=1}^{n} \nu(E_j)}$$

for some support function $\nu : 2^\Omega \to \mathbb{R}^+$ such that $\sum_{E_j \in \pi} \nu(E_j) > 0$.

The required axioms are

**Axiom 8** (The Sure-Thing Principle (STP)). For all acts $f, g, h, h' \in \mathcal{H}$ and events $E \subset \Omega$,

$$fEh \succeq gEh \Rightarrow fEh' \succeq gEh'.$$

Each composite act $fEh$ equals $f$ and $h$ contingent on $E$ and $E^c$ respectively.

**Axiom 9** (Binary Bet Acyclicity (BBA)). For any cycle of events $E_1, E_2, \ldots, E_n, E_1$ such that $E_1 \cap E_2 = E_2 \cap E_3 = \cdots = E_{n-1} \cap E_n = E_n \cap E_1 = \emptyset$, and for any outcomes $x_1, \ldots, x_n, y \in X$,

$$x_1E_1y \succ x_2E_2y \succ \cdots \succ x_nE_ny \Rightarrow x_1E_1y \succeq x_nE_ny.$$  

The same axioms can be applied to the weak preference $\succeq$ in my model.

**Theorem 6.** A monotonic preference $\succeq$ satisfies CPT, PM, CI, CA, PB, STP, and BBA if and only if it is represented by (22), where $u \in \mathcal{U}$ and the belief function $q : \Pi \to \Delta$ has the form (24) for some $\nu : 2^\Omega \to \mathbb{R}^+$ such that $\sum_{E_j \in \pi} \nu(E_j) > 0$ for all $\pi \in \Pi$.

**Proof.** Suppose that $\succeq$ satisfies the required axioms. By Theorem 4, it has representation (22). Define $\succeq^*$ via (23). Then it satisfies Ahn and Ergin’s model. By their Theorem 3, the belief function $q$ has the required structure.  

This result characterizes a model where the beliefs $b(f, g) = q(\pi f \lor \pi g)$ are framed by the generated cross-partition $\pi f \lor \pi g$, and the framing complies with Tversky and Koehler’s support theory.

The BPMEU and BPEU models can be extended to a more general setting where preferences $\succeq$ are defined over pairs $(h, f) \in \mathcal{H} \times \mathcal{D}$ of acts $h$ and verbal descriptions $d \in \mathcal{D}$. In this setting, the BPMEU model has the form

$$(f, d) \succeq (g, e) \iff \min_{q \in \mathcal{M}(d,e)} u(f(q)) \geq \min_{q \in \mathcal{M}(e,d)} u(g(q))$$

for all $(f, d), (g, e) \in \mathcal{H} \times \mathcal{D}$. Axioms 1–5 can be adapted accordingly. Note that mixtures of pairs $(f, d)$ and $(g, e)$ are required only when $d = e$. Representations (25) and their various special cases are studied as a separate research project.
4.3 Comparative Ignorance and Separable Events

According to Fox and Tversky [14], comparative ignorance can be revealed through comparisons of arbitrary prospects \( f \in \mathcal{H} \) with bets on events that the DM feels *competent* to evaluate. Such events can be endogenously identified via separable partitions.

Say that an event \( E \subset \Omega \) is *separable* if \( E \in \theta \) for some separable partition \( \theta \in \Pi_s \). Let \( \Sigma_s \) be the collection of all separable events.

Suppose that \( \succeq \) has a BPMEU representation (13) with a function \( M : \Pi \times \Pi \to \mathcal{M} \). Then \( \Omega \in \Sigma_s \) is separable. By convention, let \( \emptyset \in \Sigma_s \) as well.

Theorem 5 implies further that \( \Omega_s \) is a \( \lambda \)-system: for all separable events \( E, E' \in \Sigma_s \),

- \( E^c = \Omega \setminus E \in \Sigma_s \),
- if \( E \) and \( E' \) are disjoint, then \( E \cup E' \in \Sigma_s \).

To show this claim, take a probability measure \( q^* \in \Delta \) such that \( M_{\theta}(\pi, \tau) = \{q_{\theta}^*\} \) for all partitions \( \pi, \tau \in \Pi \) and resolute partitions \( \theta \in \Sigma \). Fix disjoint separable events \( E, E' \in \Sigma_s \) and partitions \( \pi, \tau \in \Pi \). Then for all \( q \in M(\pi, \tau), q(E) = q^*(E) \) and \( q(E') = q^*(E') \). Thus \( q(E^c) = 1 - q^*(E) = q^*(E^c) \) and \( q(E \cup E') = q^*(E) + q^*(E') = q^*(E \cup E') \). As \( \pi, \tau \in \Pi \) are arbitrary, then the events \( E^c \) and \( E \cup E' \) are both separable.\(^5\)

The invariance of the DM’s beliefs over separable events suggests that she should feel *competent* to evaluate a binary bet \( xEy \) when \( E \) is separable, but find it more difficult to evaluate other non-constant acts \( f \) with non-separable partitions \( \pi f \). This contrast motivates the following definition of comparative ignorance.

Say that \( \succeq \) exhibits *comparative ignorance* if for all \( x, y \in X, g \in \mathcal{H}, \) and separable events \( E \in \Sigma_s \),

\[
    x \succ y \succeq g \quad \Rightarrow \quad xEy \succeq g.
\]

This definition requires that the evaluation of any act \( g \in \mathcal{H} \) cannot improve when the alternative constant \( y \in X \) is replaced by a binary bet \( xEy \) on some separable event \( E \). Note that the value of the bet \( xEy \) should always exceed \( y \). Thus \( xEy \) should be chosen over \( g \) whenever \( y \) is weakly better than \( g \).

To characterize this behavioral notion, it is convenient to assume the monotonic CPMEU model.\(^6\)

\(^5\)Note that the disjoint union of resolute events \( E \subset \Omega \) such that \( E \in \theta \) for some \( \theta \in \Pi_r \) need not be resolute. Moreover, a partition \( \pi \) that consists of resolute events need not be resolute either. Thus the collection of resolute events need not be a \( \lambda \)-system or even a mosaic. Kopylov [23] establishes that endogenous definitions of unambiguous events in Savage’s setting imply the mosaic structure.

\(^6\)Comparative ignorance can be characterized in the general BPMEU specification as well.
Theorem 7. If \( \succeq \) satisfies CPT, Axioms 2–5, and monotonicity, then the following claims are equivalent:

(i) \( \succeq \) exhibits comparative ignorance,

(ii) \( \succeq \) is represented by

\[
      f \succeq g \iff \min_{q \in M(\pi f \lor \pi g)} u(f(q)) \geq \min_{q \in M(\pi g \lor \pi f)} u(g(q)) \quad (27)
\]

where \( M(\pi) \subset M(\pi \lor \theta) \) for all \( \pi \in \Pi \) and separable partitions \( \theta \in \Pi_s \).

Here the expansion condition \( M(\pi) \subset M(\pi \lor \theta) \) captures the (weakly) increasing aversion towards ambiguous events in any partition \( \pi \in \Pi \) when \( \pi \) is crossed with an arbitrary separable partition \( \theta \in \Pi_s \).

4.4 Choice in Non-Binary Sets

The CPMEU and CPEU models have natural extensions for choices in non-binary sets.

Let \( \mathcal{M} \) be the set of all menus—non-empty finite subsets \( F \subset \mathcal{H} \). For each \( F \in \mathcal{M} \), let \( \pi F = \pi f_1 \lor \pi f_2 \lor \cdots \lor \pi f_n \) be the cross-partition generated by all feasible acts in a menu \( F = \{f_1, \ldots, f_n\} \).

A choice function \( c : \mathcal{M} \to \mathcal{M} \) specifies a non-empty set \( c(F) \subset F \) of all acts that the DM may be willing to choose in a given menu \( F \in \mathcal{M} \). Suppose that \( c(\cdot) \) is given as a primitive. Then CPEU and CPMEU representations can be adapted for the function \( c(\cdot) \) as follows: for all \( F \in \mathcal{M} \),

\[
      c(F) = \arg \max_{f \in F} u(f(p(\pi f, \pi F))),
      c(F) = \arg \max_{f \in F} \min_{q \in M(\pi f \lor \pi F)} u(f(q)) \quad (28)
\]

where \( u \in \mathcal{U}, p : \Pi \times \Pi \to \Delta, \) and \( M : \Pi \times \Pi \to \mathcal{M} \) respectively.

Define the base relation \( \succeq \) for the choice function \( c(\cdot) \) as

\[
      f \succeq g \iff f \in c\{f, g\}.
\]

Then (28) implies that \( \succeq \) satisfies the CPEU and CPMEU models respectively. Note that the general BPEU and BPMEU models do not have any obvious extensions of this sort.

4.5 Axioms via Bivariate Beliefs

It is instructive to derive axioms from the assumptions imposed on the bivariate beliefs in the BEU model. Suppose that \( \succeq \) is represented by

\[
      f \succeq g \iff u(f(b(f, g))) \geq u(g(b(g, f))) \iff f(b(f, g)) \succeq g(b(g, f))
\]
for some \( u \in \mathcal{U} \) and \( b : \mathcal{H} \times \mathcal{H} \to \Delta \). RM and regularity are obvious.

If \( b(f, g) = b(f, f) \) for all \( f, g \in \mathcal{H} \), then \( \succeq \) has a utility representation \( U(f) = u(f(b(f, f))) \) and hence, satisfies transitivity. Suppose that \( b(f, g) = b(g, f) \) for all \( f, g \in \mathcal{H} \) in the BEU model. Take any \( f, g \in \mathcal{H} \) and let \( p = b(f, g) \). If \( f \succeq g \), then \( u(f(p)) \geq u(g(p)) \) and hence, \( f \succeq g \). Thus \( \succeq \) is monotonic.

Assume that \( \succeq \) is represented by the BPMEU model (13) with \( M : \Pi \times \Pi \to \mathcal{M} \). Take any acts \( f, g \in H \). For any \( \tau \in \Pi \), fix a belief

\[
b^*(f, \tau) \in \arg \min_{q \in M(\pi f, \tau)} u(f(q)).
\]

Assign \( b(f, g) = b^*(f, \pi g) \). Then the beliefs \( b(f, g) \) satisfy B1 and B2 because

(B1) \( b(f, g) = b(f, g') = b(f, \pi g) \) for all \( g' \in \mathcal{A}_g \),

(B2) \( f(b(f, g)) \preceq f(b(f', g)) \) because \( b(f', g) \in M(\pi f, \pi g) \) for all \( f' \in \mathcal{A}_f \).

By (12), B1 implies PT. B1 and B2 together imply PM, CI, CA, and PUA as well.

**PM:** Take any \( f, g \in \mathcal{H} \), \( f' \in \mathcal{A}_f \) and \( g' \in \mathcal{A}_g \). By B1 and B2, if \( f \succeq f' \succeq g' \geq g \), then

\[
f(b(f, g)) \succeq f'(b(f', g)) \succeq f'(b(f', g')) \succeq g'(b'(f', f')) = b(g', f) \succeq g(b(g', f)) \succeq g(b(g, f))
\]

and hence, \( f \succeq g \).

**CI and CA:** Take any \( f, g \in \mathcal{H} \) and \( x \in X \). For all \( \alpha \in (0, 1) \), let \( f_\alpha = \alpha f + (1 - \alpha)x \) and \( g_\alpha = \alpha g + (1 - \alpha)x \). Let \( p = b(f, g) \), \( q = b(g, f) \), \( p_\alpha = b(f_\alpha, g_\alpha) \), and \( q_\alpha = b(g_\alpha, f_\alpha) \). By B1, \( p_\alpha = b(f, g) \) and \( q_\alpha = b(g, f) \). By B2, \( f(p) \preceq f(p_\alpha) \) and \( f_\alpha(p_\alpha) \preceq f_\alpha(p) \). Thus \( f(p) \sim f(p_\alpha) \). Similarly, \( g(q) \sim g(q_\alpha) \). By (8),

\[
f \succeq g \iff f(p) \succeq g(q) \iff f(p_\alpha) \succeq g(q_\alpha) \iff f_\alpha(p_\alpha) \succeq g_\alpha(q_\alpha) \iff f_\alpha(p_\alpha) \succeq g_\alpha(p_\alpha),
\]

and CI holds. Moreover, if \( f(p) \succ g(q) \), then \( \alpha f(p_\alpha) + (1 - \alpha)x \succ (\sim)g(q) \) and \( f(p) \succ \beta g(q) \) for some \( \alpha, \beta \in (0, 1) \). Thus CA holds.

**PUA:** Take any \( f, h \in \mathcal{H} \), \( g \in \mathcal{A}_f \), and \( \alpha \in [0, 1] \) such that \( f_\alpha = \alpha f + (1 - \alpha)g \in \mathcal{A}_f \). Suppose that \( f \succeq h \) and \( g \succeq h \). Let \( p_1 = b(h, f) \), \( p_\alpha = b(f_\alpha, h) \), \( p_0 = b(g, h) \). By B1, \( b(h, f) = b(h, f_\alpha) = q \). By B2, \( f(p_\alpha) \succeq f(p_1) \) and \( g(p_\alpha) \succeq g(p_0) \). Then

\[
f_\alpha(p_\alpha) = \alpha f(p_\alpha) + (1 - \alpha)g(p_\alpha) \succeq \alpha f(p_1) + (1 - \alpha)g(p_0) \succeq \alpha h(q) + (1 - \alpha)h(q) = h(q).
\]

Thus \( f_\alpha \succeq h \).

Suppose next that \( \succeq \) is represented by the BPEU model (14) with the belief function \( p : \Pi \times \Pi \to \Delta \). For any acts \( f, g \in H \), let \( b(f, g) = p(\pi f, \pi g) \). Then \( b \) satisfies
Figure 1: Refinements of the BEU Model

(B3) \( b(f, g) = b(f', g') \) for all \( f' \in A_f \) and \( g' \in A_g \).

B3 implies B1 and B2, and hence Axioms 1–5.

Moreover, B3 implies PB.

PB: Take any \( f, h \in H, g \in A_f \), and \( \alpha \in [0, 1] \) such that \( f_\alpha = \alpha f + (1 - \alpha)g \in A_f \). By B3, \( b(f, h) = b(g, h) = b(f_\alpha, h) \) and \( b(h, f) = b(h, g) = b(h, f_\alpha) \). Denote these measures by \( p \) and \( q \) respectively. Suppose that \( h \succeq f \) and \( h \succeq g \). Then \( h(q) \succeq f(p) \) and \( h(q) \succeq g(p) \) imply that \( h(q) \succeq f_\alpha(p) \). Thus \( h \succeq f_\alpha \).

Finally, suppose that \( \succeq \) is represented by the CPMEU model (16). For any \( \tau \in \Pi \), fix a belief

\[
b^*(f, \tau) \in \arg \min_{q \in M(\pi f, \pi f \lor \tau)} u(f(q)).
\]

Assign \( b(f, g) = b^*(f, \pi f \lor \pi g) \) for all \( f, g \in H \). Then \( b \) satisfies B1*, which in turn implies CPT.

CPEU is the overlap of BPEU and CPMEU and hence, satisfies all of the above axioms.

4.6 Summary

Figure 1 provides a map of the various refinements of the BEU model that are obtained in Theorems 1–6.

A APPENDIX: PROOFS

The central part of the proofs is the construction of the BPMEU model (13) in Theorem 2. I do this construction first, and then proceed to all other results.

Suppose that \( \succeq \) satisfies Axioms 1–5 (PT, PM, CI, CA, PUA). Then the restriction of \( \succeq \) to \( X \) satisfies transitivity, Independence, and Archimedean. Thus
≥ is represented on X by a linear utility index \( u : X \rightarrow \mathbb{R} \) (Theorem 5.4 in Kreps [25]). If \( u \) is constant, then by RM, \( \succ \) is empty. Hence \( u \in \mathcal{U} \) is not constant.

For any \( f, g \in \mathcal{H} \), write \( f \gg g \) if \( f(\omega) \succ g(\omega') \) for all \( \omega, \omega' \in \Omega \).

Suppose that \( f \gg g \). Take \( y, z \in X \) such that

\[
\min_{\omega \in \Omega} u(f(\omega)) > u(y) > u(z) > \max_{\omega \in \Omega} u(g(\omega)).
\]

Then \( f \gg y \succ z \gg g \). By PM, \( f \succ g \). Moreover, for any \( \tau \in \Pi \),

\[
f \gg h_\tau \gg g
\]

for some \( h_\tau \in \mathcal{A}_\tau \). (29)

Such \( h_\tau \in \mathcal{H} \) can be defined for \( \tau = \{E_1, \ldots, E_k\} \) as

\[
h_\tau(\omega) = \frac{1}{i+1} y + \frac{i}{i+1} z \quad \text{for all } i = 1, \ldots, k \text{ and } \omega \in E_i.
\]

Wlog, assume that \([-1, 1] \subset u(X)\). Take \( c_0 \in X \) such that \( u(c_0) = 0 \).

Take any partitions \( \pi, \tau \in \Pi \). For all \( f \in \mathcal{H} \), let

\[
V(f, \pi, \tau) = \sup_{(\alpha, z) \in Z(f, \pi, \tau)} \frac{u(z)}{\alpha}
\]

where \( Z(f, \pi, \tau) \) is the set of all pairs \((\alpha, z) \in (0, 1] \times X\) such that

\[
\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z \quad \text{for some } f_\pi \in \mathcal{A}_\pi \text{ and } f_\tau \in \mathcal{A}_\tau.
\]

I claim that there is a set \( M(\pi, \tau) \in \mathcal{M} \) such that for all \( f \in \mathcal{H} \),

\[
V(f, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(f(q)).
\]

In this way, the sets \( M(\pi, \tau) \in \mathcal{M} \) can be identified for all \( \pi, \tau \in \Pi \).

Before starting the tedious proof of this claim, show that the maxmin structure (32) delivers the required BPMEU representation (13).

**Lemma 8.** If (32) holds, then for all \( f, g \in \mathcal{H} \),

\[
f \succeq g \iff V(f, \pi f, \pi g) \geq V(g, \pi g, \pi f).
\]

**Proof.** Fix \( f, g \in \mathcal{H} \). Let \( \pi = \pi f \) and \( \tau = \pi g \).

Suppose that \( f \succeq g \). Take any \((\alpha, z) \in Z(g, \tau, \pi)\). Then \( \alpha g + (1 - \alpha)c_0 \geq g_\tau \succ g_\pi \gg z \) for some \( g_\pi \in \mathcal{A}_\pi \) and \( g_\tau \in \mathcal{A}_\tau \). By (29), there is \( f_\tau \in \mathcal{A}_\tau \) such that \( g_\pi \gg f_\tau \gg z \). Let \( f_\pi = \alpha f + (1 - \alpha)c_0 \). By CI,

\[
f_\pi \geq \alpha f + (1 - \alpha)c_0 \geq \alpha g + (1 - \alpha)c_0 \geq g_\tau.
\]

By PM, \( f_\pi \succeq g_\tau \succ g_\pi \succ f_\tau \). By PT, \( f_\pi \succ f_\tau \) because \( f_\tau \succeq f_\pi \) would imply \( f_\tau \succeq g_\pi \), which is not true. Thus \((\alpha, z) \in Z(f, \pi, \tau)\) because

\[
\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z.
\]

26
The inclusion \( Z(g, \tau, \pi) \subset Z(f, \pi, \tau) \) implies \( V(g, \tau, \pi) \leq V(f, \pi, \tau) \).

Suppose that \( f \succ g \). Let \( f' = \frac{f + g}{2} \) and \( g' = \frac{g + c_0}{2} \). Take \( x \in X \) such that \( f' \gg x \). By CI, \( f' \succ g' \). By CA, \( \alpha f' + (1 - \alpha) x \succ g' \) for some \( \alpha \in (0, 1) \). By (32),

\[
\frac{1}{2} V(f, \pi, \tau) = V(f', \pi, \tau) > V(\alpha f' + (1 - \alpha)x, \pi, \tau) \geq V(g', \pi, \tau) = \frac{1}{2} V(g, \tau, \pi).
\]

Thus \( V(f, \pi, \tau) > V(g, \tau, \pi) \), which implies (33). \( \square \)

It remains to construct (32).

**Lemma 9.** The function \( V : \mathcal{H} \times \Pi \times \Pi \to \mathbb{R} \) is well-defined, and

\[ V(x, \pi, \tau) = u(x) \quad \text{for all } \pi, \tau \in \Pi \text{ and } x \in X. \]

**Proof.** Take any \( \pi, \tau \in \Pi \) and \( f, g \in \mathcal{H} \). As \( f \) has finite range, there are \( x^*, x_* \in X \) such that \( x^* \geq f \geq x_* \). Take \( y_*, z_* \in X \) such that \( 0.5x_* + 0.5c_0 \succ y_* \succ z_* \). By (29), there are \( h_\pi \in \mathcal{A}_\pi \) and \( h_\tau \in \mathcal{A}_\tau \) such that

\[ 0.5f + 0.5c_0 \geq 0.5x_* + 0.5c_0 \gg h_\pi \gg y_* \gg h_\tau \gg z_. \]

Thus \((0.5, z_*) \in Z(f, \pi, \tau)\), and \( Z(f, \pi, \tau) \) is not empty. By definition, \( V(f, \pi, \tau) \geq 2u(z_*) \). As \( u(z_*) \) is arbitrarily close to \( u(0.5x_* + 0.5c_0) \), then \( V(f, \pi, \tau) \geq u(x_*) \).

Take any \((\alpha, z) \in Z(f, \pi, \tau)\). There are \( f_\pi \in \mathcal{A}_\pi \) and \( f_\tau \in \mathcal{A}_\tau \) such that

\[ \alpha x^* + (1 - \alpha)c_0 \geq \alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \geq z. \]

By PM, \( \alpha x^* + (1 - \alpha)c_0 \geq z \) because \( f_\tau \geq z \succ \alpha x^* + (1 - \alpha)c_0 \geq f_\pi \) would imply \( f_\tau \succ f_\pi \), which is not true. Therefore

\[
V(f, \pi, \tau) = \sup_{(\alpha, z) \in Z(f, \pi, \tau)} \frac{u(z)}{\alpha} \leq \frac{u(\alpha x^* + (1 - \alpha)c_0)}{\alpha} = u(x^*).
\]

Thus \( V(f, \pi, \tau) \leq u(x^*) \), and hence, \( V(f, \pi, \tau) \) is well-defined. For any \( x \in X \), the inequalities \( u(x) \geq V(x, \pi, \tau) \geq u(x) \) imply \( V(x, \pi, \tau) = u(x) \). \( \square \)

Note that the function \( V : \mathcal{H} \times \Pi \times \Pi \to \mathbb{R} \) is monotonic in its first variable: for all \( f, g \in \mathcal{H} \) and \( \pi, \tau \in \Pi \),

\[ f \geq g \implies V(f, \pi, \tau) \geq V(g, \pi, \tau) \tag{34} \]

because \( f \geq g \) implies \( Z(f, \pi, \tau) \supset Z(g, \pi, \tau) \).

Show a general property of partitions.

**Lemma 10.** The set \( I = \{ \gamma \in (0, 1) : \pi(\gamma f + (1 - \gamma)g) \neq \pi f \vee \pi g \} \) is finite.
Proof. Fix any \( f, g \in \mathcal{H} \) and a mixture \( h = \alpha f + (1 - \alpha)g \) for \( \alpha \in (0, 1) \). Show that \( \pi f \lor \pi g = \pi h \) whenever \( \alpha \not\in I \) for some finite \( I \subset (0, 1) \).

Let \( f(\Omega) = \{x_1, \dotsc, x_n\} \) and \( g(\Omega) = \{y_1, \dotsc, y_r\} \) be the finite ranges of the acts \( f, g \) with \( x_i \neq x_j \) and \( y_i \neq y_j \) for all \( i \neq j \). Let

\[
Y_\alpha = \{ \alpha x_i + (1 - \alpha) y_j : i = 1, \dotsc, n \text{ and } j = 1, \dotsc, r \}
\]

be the set of all distinct mixtures \( \alpha x_i + (1 - \alpha) y_j \). Note that if \( Y_\alpha \) has \( nr \) elements, then \( \pi f \lor \pi g = \pi h \).

Fix any indices \( i, k \in \{1, \dotsc, n\} \) and \( j, m \in \{1, \dotsc, r\} \). Suppose that the pairs \((i, j)\) and \((k, m)\) are distinct. Then \( x^* = x_i - x_k \) and \( y^* = y_m - y_j \) are non-zero elements of the linear space \( L \) that contains \( X \). Thus the equality \( \alpha x^* = (1 - \alpha)y^* \) can hold for at most one value \( \alpha = \alpha_{ijkm} \in (0, 1) \). Let \( I \) be the finite set of all such values \( \alpha_{ijkm} \). The set \( Y_\alpha \) consists of \( nr \) elements for all \( \alpha \not\in I \).

Show two familiar properties for the function \( V \).

**Lemma 11.** For all \( \pi, \tau \in \Pi \), \( f, g \in \mathcal{H} \), \( x \in X \), and \( \gamma \in [0, 1] \),

\[
\begin{align*}
V(\gamma f + (1 - \gamma)x, \pi, \tau) & = \gamma V(f, \pi, \tau) + (1 - \gamma)u(x) \quad (35) \\
V(\gamma f + (1 - \gamma)g, \pi, \tau) & \geq \min\{V(f, \pi, \tau), V(g, \pi, \tau)\} \quad (36)
\end{align*}
\]

Proof. Take any \( \pi, \tau \in \Pi \). If \( \gamma = 0 \), then (35) follows from Lemma 9.

Take \( f \in \mathcal{H} \), \( x \in X \), and \( \gamma \in (0, 1] \). Let \( g = \gamma f + (1 - \gamma)x \). Take any \((\alpha, z) \in Z(g, \pi, \tau)\). There are \( g_\pi \in A_\pi \) and \( g_\tau \in A_\tau \) such that

\[
\alpha g + (1 - \alpha)c_0 \geq g_\pi \succ g_\tau \gg z.
\]

Take \( \beta \in (0, 1) \) and \( y \in X \) such that

\[
\beta \alpha (1 - \gamma)u(x) + (1 - \beta)u(y) = 0.
\]

Then

\[
g' \geq \beta g_\pi + (1 - \beta)y \succ \beta g_\tau + (1 - \beta)y \geq \beta z + (1 - \beta)y
\]

where \( g' = \beta(\alpha g + (1 - \alpha)c_0) + (1 - \beta)y \) satisfies \( \beta \alpha \gamma f + (1 - \beta \alpha \gamma)c_0 \geq g' \). Thus

\[
V(f, \pi, \tau) \geq \frac{u(\beta z + (1 - \beta)y)}{\beta \alpha \gamma} = \frac{\beta u(z) - \beta \alpha (1 - \gamma)u(x)}{\beta \alpha \gamma}
\]

that is, \( \gamma V(f, \pi, \tau) + (1 - \gamma)u(x) \geq \frac{u(z)}{\alpha} \). As \( (\alpha, z) \in Z(g, \pi, \tau) \) is arbitrary, then

\[
\gamma V(f, \pi, \tau) + (1 - \gamma)u(x) \geq V(g, \pi, \tau).
\]

Conversely, suppose that

\[
\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z.
\]

28
for some \((\alpha, z) \in Z(f, \pi, \tau)\), \(f_\pi \in \mathcal{A}_\pi\), and \(f_\tau \in \mathcal{A}_\tau\). Then
\[\alpha g + (1 - \alpha)c_0 \geq g_\pi \succ g_\tau \gg y,\]
where
\[g_\pi = \gamma f_\pi + (1 - \gamma)(\alpha x + (1 - \alpha)c_0)\]
\[g_\tau = \gamma f_\tau + (1 - \gamma)(\alpha x + (1 - \alpha)c_0)\]
\[y = \gamma z + (1 - \gamma)(\alpha x + (1 - \alpha)c_0).\]
Thus \(V(g, \pi, \tau) \geq \frac{u(y)}{\alpha} = \gamma \frac{u(z)}{\alpha} + (1 - \gamma)u(x)\), and
\[V(g, \pi, \tau) \geq \gamma V(f, \pi, \tau) + (1 - \gamma)u(x)\]
holds because \((\alpha, z) \in Z(f, \pi, \tau)\) is arbitrary. Thus (35) holds.

Turn to (36). I claim that the function
\[v(\alpha) = V(\alpha f + (1 - \alpha)g, \pi, \tau)\]
is continuous for all \(\alpha \in [0, 1]\). To show this claim, fix \(x^*, x_*, y^*, y_* \in X\) such that \(x^* \geq f \geq x_*\) and \(y^* \geq g \geq y_*\). By (34) and (35)
\[\alpha u(x^*) + (1 - \alpha)V(g, \pi, \tau) = V(\alpha x^* + (1 - \alpha)g) \geq v(\alpha) \geq V(\alpha x_* + (1 - \alpha)g) \geq \alpha u(x_*) + (1 - \alpha)V(g, \pi, \tau).\]
Thus, \(v\) is continuous at \(\alpha = 0\). Fix any \(\alpha \in (0, 1)\). Let \(h = \alpha f + (1 - \alpha)g\).
The above argument implies that the function \(w(\beta) = V(\beta g + (1 - \beta)h, \pi, \tau)\) is continuous at \(\beta = 0\). Thus the function \(v(\alpha - \varepsilon) = w(\frac{\alpha}{\alpha - \varepsilon})\) is continuous at \(\varepsilon = 0\).
To show continuity of \(v(\alpha + \varepsilon)\) at \(\varepsilon = 0\), switch \(f\) with \(g\), and \(\alpha\) with \(1 - \alpha\). Use the same switch to show that \(v(\alpha)\) is continuous at \(\alpha = 1\). Thus, \(v(\alpha)\) is continuous at any \(\alpha \in [0, 1]\).

Proceed to show (36). Take any \(f, g \in \mathcal{H}, \pi, \tau \in \Pi\), and \(\lambda \in \mathbb{R}\) such that
\[\lambda < \min\{V(f, \pi, \tau), V(g, \pi, \tau)\}.\]
By definition of \(V\), there are pairs \((\alpha, z_f) \in Z(f, \pi, \tau)\) and \((\beta, z_g) \in Z(g, \pi, \tau)\), and acts \(f_\pi, g_\pi \in \mathcal{A}_\pi\) and \(f_\tau, g_\tau \in \mathcal{A}_\tau\) such that
\[\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z_f \quad \text{and} \quad \alpha \lambda < u(z_f)\]
\[\beta g + (1 - \beta)c_0 \geq g_\pi \succ g_\tau \gg z_g \quad \text{and} \quad \beta \lambda < u(z_g).\]
Wlog \(\beta \geq \alpha\). Let \(g_\pi = \frac{\alpha}{\beta}g_\pi + \frac{\beta - \alpha}{\beta}c_0\), \(g_\tau = \frac{\alpha}{\beta}g_\tau + \frac{\beta - \alpha}{\beta}c_0\), and \(z_g = \frac{\alpha}{\beta}z_g + \frac{\beta - \alpha}{\beta}c_0\). Then
\[\alpha g + (1 - \alpha)c_0 \geq g_\pi' \succ g_\tau' \gg z_g' \quad \text{and} \quad \alpha \lambda = \frac{\alpha}{\beta} \beta \gamma < \frac{\alpha}{\beta} u(z_g) = u(z_g').\]
Let \(z \in X\) be the worse of the two lotteries \(z_f\) and \(z_g\). Then
\[\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z\]
\[\alpha g + (1 - \alpha)c_0 \geq g_\pi' \succ g_\tau' \gg z,\]
and \(\alpha \lambda < u(z)\). Consider two cases.
(i) $f_\tau \succeq g'_\tau$. By PT, $f_\pi \succeq f_\tau \succeq g'_\tau$ implies that $f_\pi \succeq g'_\tau$. Take any $\gamma \in [0, 1]$ such that $\gamma f_\pi + (1 - \gamma) g'_\pi \in \mathcal{A}_\pi$. By PUA,

$$\gamma f_\pi + (1 - \gamma) g'_\pi \succeq g'_\tau.$$ 

Take $h_\pi \in \mathcal{A}_\pi$ and $h_\tau \in \mathcal{A}_\tau$ such that $g'_\tau \gtrsim h_\pi \gtrsim h_\tau \gtrsim z$.

If $h_\tau \succeq \gamma f_\pi + (1 - \gamma) g'_\pi$, then

$$h_\tau \succeq \gamma f_\pi + (1 - \gamma) g'_\pi \succeq g'_\tau \succeq h_\pi,$$

implies by PT that $h_\tau \succeq h_\pi$. Thus $\gamma f_\pi + (1 - \gamma) g'_\pi \succeq h_\tau$. As

$$\alpha(\gamma f + (1 - \gamma)g) + (1 - \alpha)c_0 \geq \gamma f_\pi + (1 - \gamma) g'_\pi \succeq h_\tau \gtrsim z,$$

then $V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \frac{u(z)}{\alpha} > \lambda$.

(ii) $g'_\tau \succeq f_\tau$. Take any $\gamma \in [0, 1]$ such that $\gamma f_\pi + (1 - \gamma) g'_\pi \in \mathcal{A}_\pi$. By PUA,

$$\gamma f_\pi + (1 - \gamma) g'_\pi \succeq f_\tau.$$ 

Then by (29), there are $h_\pi \in \mathcal{A}_\pi$ and $h_\tau \in \mathcal{A}_\tau$ such that

$$f_\tau \gtrsim h_\pi \gtrsim h_\tau \gtrsim z.$$ 

If $h_\tau \succeq \gamma f_\pi + (1 - \gamma) g'_\pi$, then

$$h_\tau \succeq \gamma f_\pi + (1 - \gamma) g'_\pi \succeq f_\tau \succeq h_\pi,$$

and by PT, $h_\tau \succeq h_\pi$. Yet $h_\pi \gtrsim h_\tau$. Thus $\gamma f_\pi + (1 - \gamma) g'_\pi \succeq h_\tau$. As

$$\alpha(\gamma f + (1 - \gamma)g) + (1 - \alpha)c_0 \geq \gamma f_\pi + (1 - \gamma) g'_\pi \succeq h_\tau \gtrsim z,$$

then $V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \frac{u(z)}{\alpha} > \lambda$.

In each case, the inequality $V(\gamma f + (1 - \gamma)g, \pi f, \pi g) > \lambda$ holds for any $\gamma$ such that $\gamma f_\pi + (1 - \gamma) g'_\pi \in \mathcal{A}_\pi$. By Lemma 10, this inclusion holds for all $\gamma \in [0, 1] \setminus I$ where $I$ is a finite set. By continuity, the function

$$v(\gamma) = V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \lambda,$$

for all $\gamma \in [0, 1]$. As $\lambda \in \mathbb{R}$ is arbitrarily close to $\min\{V(f, \pi, \tau), V(g, \pi, \tau)\}$, then (36) holds.

The above lemma implies that for any partitions $\pi, \tau \in \Pi$, the preference $\succeq^*$ represented by $V^*(f) = V(f, \pi, \tau)$ on $\mathcal{H}$ must satisfy all conditions of the multiple priors model in Gilboa and Schmeidler [17]. Moreover, for all $x \in X$, $V^*(x) = u(x)$. By Gilboa–Schmeidler’s Theorem 1, there is a unique set $M(\pi, \tau) \in \mathcal{M}$ such that (32) holds. Lemma 8 implies the required BPMEU representation (13).
Properties of the sets \( M(\pi, \tau) \)

The combination of (32) and (33) requires that for all \( f, g, h \in \mathcal{H} \) and \( \pi, \tau \in \Pi \),

\[
\begin{align*}
  f \succeq g & \iff V(f, \pi f, \pi g) \geq V(g, \pi g, \pi f) \\
  V(h, \pi, \tau) &= \min_{q \in M(\pi, \tau)} u(h(q)) \tag{38}
\end{align*}
\]

where \( u \in \mathcal{U}, M : \Pi \times \Pi \rightarrow \mathcal{M} \), and \( V : \mathcal{H} \times \Pi \times \Pi \rightarrow \mathbb{R} \) is defined via (30).

Moreover, \( u \) is unique up to plt, and the function \( M \) is unique.

Lemma 12. For all \( \pi, \tau \in \Pi \),

(i) \( M(\pi, \tau) = \{ q \in \Delta : q_\pi \in M_\pi(\pi, \tau) \} \);

(ii) if \( \succeq \) satisfies PB, then \( M_\pi(\pi, \tau) \) is a singleton,

(iii) if \( \succeq \) is monotonic, then \( M_\pi(\pi, \tau) \subseteq M_\pi(\tau, \pi) \),

(iv) if \( \succeq \) is transitive, then \( M(\pi, \tau) = M(\pi, \pi) \),

(v) if \( \succeq \) satisfies CPT, then \( M(\pi, \tau) = M(\pi, \pi \lor \tau) \).

Proof. Fix any \( \pi, \tau \in \Pi \). Suppose that \( \pi = \{ E_1, \ldots, E_n \} \). Identify each element \( q_\pi \in \Delta_\pi \) with a vector \((q_\pi(E_1), \ldots, q_\pi(E_n)) \in \mathbb{R}^n\).

Show (i). Let \( P(\pi, \tau) = \{ q \in \Delta : q_\pi \in M_\pi(\pi, \tau) \} \). By definition, the set \( P(\pi, \tau) \) is convex and closed. Take any \( p \in P(\pi, \tau) \). Then \( p_\pi = p_\pi^* \) for some \( p^* \in M(\pi, \tau) \).

Take any \( f \in \mathcal{H} \) and \((\alpha, z) \in Z(f, \pi, \tau)\). Then

\[
\alpha f + (1 - \alpha) c_0 \succeq f_\pi \succ f_\tau \succ z
\]

for some \( f_\pi \in \mathcal{A}_\pi \) and \( f_\tau \in \mathcal{A}_\tau \). By (38),

\[
\alpha u(f(p)) \geq u(f_\pi(p)) = u(f_\pi(p^*)) \geq V(f_\pi, \pi, \tau) > V(f_\tau, \tau, \pi) > u(z).
\]

Thus \( u(f(p)) > \frac{u(z)}{\alpha} \) for all \((\alpha, z) \in Z(f, \pi, \tau)\). By (30), \( u(f(p)) \geq V(f, \pi, \tau) \). As \( p \in P(\pi, \tau) \) is arbitrary, then

\[
\min_{q \in P(\pi, \tau)} u(f(q)) \geq V(f, \pi, \tau).
\]

Yet \( V(f, \pi, \tau) \geq \min_{q \in P(\pi, \tau)} u(f(q)) \) because \( M(\pi, \tau) \subseteq P(\pi, \tau) \). Thus

\[
\min_{q \in P(\pi, \tau)} u(f(q)) = V(f, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(h(q))
\]

and hence, \( P(\pi, \tau) = M(\pi, \tau) \) because both sets are convex and closed.
Turn to (ii). Suppose that $\succeq$ satisfies PB, and $M_\pi(\pi, \tau)$ is not a singleton. Then there are $p, q \in M(\pi, \tau)$ such that $p_\pi \neq q_\tau$. Wlog, assume that $p(E_1) > q(E_1)$. Take $\alpha, \beta \in (0, 1)$ such that

$$p(E_1) > \frac{\alpha}{\alpha + \beta} > q(E_1) \quad \text{and} \quad q(E_1^c) > \frac{\beta}{\alpha + \beta} > p(E_1^c).$$

Take $c_\alpha, c_\beta, z \in X$ such that $u(c_\alpha) = \alpha$, $u(c_\beta) = \beta$, and $u(z) = \frac{\alpha \beta}{\alpha + \beta}$. Take $f, g \in \mathcal{H}_\pi$ such that $f(\omega) = c_\beta$ and $g(\omega) = c_\alpha$ for $\omega \in E_1$. Let $\gamma = \frac{\alpha}{\alpha + \beta}$ and $f' = \gamma f + (1 - \gamma)g$. Then $V(f, \pi, \tau) \leq \beta q(E_1) < u(z)$ and $V(g, \pi, \tau) \leq \alpha p(E_1) < u(z)$, but $V(f', \pi, \tau) = u(z)$. Take $x, y \in X$ such that $V(f, \pi, \tau) < u(x) < u(y) < u(z)$ and $V(g, \pi, \tau) < u(x) < u(y) < u(z)$. Take $h_\tau \in \mathcal{A}_\tau$ such that $y \gg h_\tau \gg z$. Take any $h_\pi \in \mathcal{A}_\pi$. Then for all sufficiently small $\varepsilon > 0$,

$$V(\varepsilon h_\pi + (1 - \varepsilon)f, \pi, \tau) < u(x) < V(h_\tau, \pi),$$

$$V(\varepsilon h_\pi + (1 - \varepsilon)g, \pi, \tau) < u(x) < V(h_\tau, \pi),$$

$$V(\varepsilon h_\pi + (1 - \varepsilon)f', \pi, \tau) > u(y) > V(h_\tau, \pi).$$

Note that $\pi f$, $\pi g$, and $\pi f'$ are all coarser than $\pi$. By Lemma 10, $\varepsilon$ can be taken so that all the mixtures $\varepsilon h_\pi + (1 - \varepsilon)f$, $\varepsilon h_\pi + (1 - \varepsilon)g$, and $\varepsilon h_\pi + (1 - \varepsilon)f'$ generate the partition $\pi = \pi h_\pi$. The rankings $h_\tau \succ \varepsilon h_\pi + (1 - \varepsilon)f$, $h_\tau \succ \varepsilon h_\pi + (1 - \varepsilon)g$, and $h_\tau \prec \varepsilon h_\pi + (1 - \varepsilon)f'$ violate PB.

Turn to (iii). Suppose that $\succeq$ is monotonic. Take any $f \in \mathcal{A}_\pi$ and $(\alpha, z) \in Z(f, \tau, \pi)$. Then

$$\alpha f + (1 - \alpha)c_0 \geq f_\tau \succ f_\pi \gg z \quad \text{for some } f_\pi \in \mathcal{A}_\pi \text{ and } f_\tau \in \mathcal{A}_\tau.$$

Take $f'_\pi = \alpha f + (1 - \alpha)c_0$ and $f'_\tau \in \mathcal{A}_\tau$ such that

$$f'_\pi \geq f_\tau \succ f_\pi \gg f'_\tau \gg z.$$ 

Suppose that $f'_\tau \succeq f'_\pi$. As $\succeq$ is monotonic, then

$$f'_\tau \succeq f'_\pi \succeq f_\tau \succ f_\pi.$$ 

By PT, $f'_\tau \succeq f_\pi$, which contradicts $f_\pi \succ f'_\tau$. Thus $f'_\pi \succ f'_\tau$, and

$$\alpha f + (1 - \alpha)c_0 \geq f'_\pi \succ f'_\tau \gg z.$$ 

Thus $(\alpha, z) \in Z(f, \pi, \tau)$ and

$$V(f, \pi, \tau) \geq V(f, \tau, \pi).$$

Let $A = M_\pi(\pi, \tau) \subset \mathbb{R}^n$ and $B = M_\pi(\tau, \pi) \subset \mathbb{R}^n$. Both $A$ and $B$ are closed and convex sets. Suppose that $A \setminus B \neq \emptyset$. Take $a \in A \setminus B$. By the separation theorem, there is a vector $b \in \mathbb{R}^n$ such that

$$a \cdot b < \min_{c \in B} c \cdot b.$$
By continuity, $b$ can be taken so that $b_i \neq b_j$ for all $i, j \in \{1, \ldots, n\}$. Rescale $b$ so that $b_i \in [-1, 1]$ for all $i$. Define an act $g$ such that $u(g(\omega)) = b_i$ for all $i$ and $\omega \in E_i$. Then $g \in \mathcal{A}_\pi$ and

$$V(g, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(g(q)) = a \cdot b < \min_{c \in B} c \cdot b = \min_{q \in M(\pi, \tau)} u(g(q)) = V(g, \tau, \pi).$$

This contradiction with (39) shows that $A \subseteq B$, that is, $M_\pi(\pi, \tau) \subseteq M_\pi(\tau, \pi)$.

Turn to (iv). Suppose that $\succeq$ is transitive. Take any $f \in \mathcal{H}$ and $(\alpha, z) \in Z(f, \pi, \tau)$. Then

$$\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f_\tau \succ z \quad \text{for some } f_\pi \in \mathcal{A}_\pi \text{ and } f_\tau \in \mathcal{A}_\tau.$$ 

Take $f'_\pi \in \mathcal{A}_\pi$ such that $f_\pi \succ f'_\pi \succ z$. By transitivity, $f_\pi \succ f'_\pi$. Thus $(\alpha, z) \in Z(f, \pi, \pi)$ and hence, $V(f, \pi, \pi) \geq V(f, \pi, \tau)$. Similarly, if

$$\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f'_\pi \succ z \quad \text{for some } f_\pi, f'_\pi \in \mathcal{A}_\pi,$$

then there is $f_\tau \in \mathcal{A}_\tau$ such that $\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f_\tau \succ z$. Thus $V(f, \pi, \tau) \geq V(f, \pi, \pi)$. The equality $V(f, \pi, \tau) = V(f, \pi, \pi)$ for all $f \in \mathcal{H}$ implies that $M(\pi, \tau) = M(\pi, \pi)$.

Turn to (v). Suppose that $\succeq$ satisfies CPT. Take any partitions $\pi, \tau \in \Pi$ and an act $f \in \mathcal{H}_\pi$. Take $x, y \in X$ such that $V(f, \pi, \tau) = u(x)$ and $V(f, \pi, \pi \cup \tau) = u(y)$.

Suppose that $x \succ y$. Take $z \in X$ such that $x \succ z \succ y$. Take $h \in \mathcal{A}_\tau$ and $h' \in \mathcal{A}_{\pi \cup \tau}$ such that

$$x \gg h \gg z \gg h' \gg y.$$ 

Then $V(f, \pi, \tau) > V(h, \pi, \tau)$ and $V(f, \pi, \pi \cup \tau) < V(h', \pi \cup \tau, \pi)$. By CPT, the rankings $h' \succeq h' \succ f \succ h$ imply $h' \succeq h$, which contradicts $h \gg h'$.

Suppose that $y \succ x$. Take $z \in X$ such that $y \succ z \succ x$. Take $h \in \mathcal{A}_\tau$ and $h' \in \mathcal{A}_{\pi \cup \tau}$ such that

$$y \gg h' \gg z \gg h \gg x.$$ 

Then $V(f, \pi, \tau) < V(h, \pi, \tau)$ and $V(f, \pi, \pi \cup \tau) > V(h', \pi \cup \tau, \pi)$. By CPT, the rankings $h \succ f \succ h' \succeq h'$ imply $f \succeq h'$, which contradicts $h' \gg h$.

Thus $y \sim x$, that is, $V(f, \pi, \tau) = V(f, \pi, \pi \cup \tau)$. By Lemma 10, for any $\pi$-measurable $g \in \mathcal{H}_\pi$, the mixture $\alpha f + (1 - \alpha)g \in \mathcal{A}_\pi$ for all sufficiently small $\alpha$. By continuity of $V$, $V(g, \pi, \tau) = V(g, \pi, \pi \cup \tau)$ for all $\pi$-measurable acts $g \in \mathcal{H}_\pi$. The uniqueness of the multiple priors representations implies that $M_\pi(\pi, \tau) = M_\pi(\pi, \pi \cup \tau)$. \hfill \Box

### A.1 Proofs of Theorems 1–7

Theorem 1 is straightforward. Suppose that $\succ$ is not empty, $\succeq$ is complete, regular, and satisfies RM. Let $u : \mathcal{L} \to \mathbb{R}$ be a linear representation for $\succeq$ on $X$. If $u$ is constant, then by RM, $\succ$ is empty. Hence $u \in \mathcal{U}$ is not constant.
Take any \( f, g \in \mathcal{H} \). Assume \( f \succ g \). By RM, there are \( \omega, \omega' \in \Omega \) such that 
\( f(\omega) \succ g(\omega') \). Let \( p(\omega) = 1 \) and \( q(\omega') = 1 \). Then (8) holds for \( b(f, g) = p \) and \( b(g, f) = q \). Assume \( f \sim g \). By RM, there are \( \omega, \omega' \in \Omega \) such that 
\( f(\omega) \succeq g(\omega) \) and \( g(\omega') \succeq f(\omega') \). Then 
\[
\alpha(u(f(\omega)) - u(g(\omega))) = (1 - \alpha)(u(g(\omega')) - u(f(\omega'))) 
\]
for some \( \alpha \in [0, 1] \). Take \( p \in \Delta \) such that \( p(\omega) = \alpha \) and \( p(\omega') = 1 - \alpha \). Then 
\( u(f(p)) = u(g(p)) \), and (8) holds for \( b(f, g) = b(g, f) = p \).

Suppose that \( \succeq \) is monotonic. Take any \( f, g \in \mathcal{H} \). Assume \( f \succ g \). By
monotonicity, there is \( \omega \in \Omega \) such that \( f(\omega) \succ g(\omega) \). Let \( b(f, g) = b(g, f) = q \) such that 
\( q(\omega) = 1 \). Assume \( f \sim g \). Let \( b(f, g) = b(g, f) = p \) as in the general case.

Suppose that \( \succeq \) is transitive and has certainty equivalents. Then for any 
\( f \in \mathcal{H} \), there is \( x \in X \) such that \( f \sim x \). By RM, there are \( \omega, \omega' \in \Omega \) such that 
\( f(\omega) \succeq x \succeq f(\omega') \). Thus there is \( q(f) \in \Delta \) such that 
\( f(q(f)) \sim x \). For all \( g \), let 
\( b(f, g) = q(f) \).

**Theorem 2**

Suppose that \( \succeq \) satisfies Axioms 1–5 (PT, PM, CI, CA, PUA). Then \( \succeq \) is represented by (38), which is equivalent to the BPMEU model (13).

To show the uniqueness claim, suppose that \( \succeq \) has another representation (13) with
components \( u^* \in \mathcal{U} \) and \( M^* : \Pi \times \Pi \to \mathcal{M} \). Then \( u^* \) is a plt of \( u \) because both \( u \) and \( u^* \) represent the same ranking of lotteries. Wlog take \( u^* = u \). Then for all \( f, g \in \mathcal{H} \),
\[
f \succeq g \iff \min_{q \in M^*(\pi_f, \pi_g)} u(f(q)) \geq \min_{q \in M^*(\pi_g, \pi_f)} u(g(q)).
\]

Suppose that \( M_\pi(\pi, \tau) \neq M_\tau^*(\pi, \tau) \) for some \( \pi, \tau \in \Pi \). The uniqueness claim
in Gilboa–Schmeidler’s Theorem 1 implies that the two utility functions
\[
U(f) = \min_{q \in M_\pi(\pi_f, \pi)} u(f(q)) \quad U^*(f) = \min_{q \in M_\tau^*(\pi_f, \pi)} u(f(q))
\]
represent distinct rankings over acts \( f \in \mathcal{A}_\pi \). Take \( f \in \mathcal{A}_\pi \) such that \( U(f) \neq U^*(f) \). Wlog \( U(f) > U^*(f) \). Take \( x, y \in X \) such that \( U(f) > u(x) > u(y) > U^*(f) \). By (29), there is \( g \in \mathcal{A}_\pi \) such that \( x \gg g \gg y \). Then
\[
\min_{q \in M(\pi_f, \pi_g)} u(f(q)) = U(f) > u(x) > \min_{q \in M(\pi_g, \pi_f)} u(g(q)) \quad \min_{q \in M^*(\pi_g, \pi_f)} u(g(q)) > u(y) > \min_{q \in M^*(\pi_f, \pi_g)} u(f(q)).
\]
Then (38) implies $f \succ g$, but (40) implies $g \succ f$. This contradiction establishes that $M_\pi(\pi, \tau) = M_\tau^*(\pi, \tau)$ for all $\pi, \tau \in \Pi$.

Suppose that $\succeq$ is monotonic and is represented by (38). For all $\pi, \tau \in \Pi$, let

$$M^*(\pi, \tau) = M^*(\tau, \pi) = M(\pi, \tau) \cap M(\tau, \pi).$$

By Lemma 12, for each $q \in M(\pi, \tau)$, there is $q^* \in M(\tau, \pi)$ such that $q^* = q_\pi$. Thus $q^* \in M(\pi, \tau)$ and $M^*(\pi, \tau)$ is not empty. It follows also that $M^*_\pi(\pi, \tau) = M_\pi(\pi, \tau)$.

Similarly, $M_\tau^*(\pi, \tau) = M_\tau(\pi, \tau)$. Thus for all $f, g \in \mathcal{H}$,

$$\min_{q \in M(\pi f, \pi g)} u(f(q)) = \min_{q \in M^*(\pi f, \pi g)} u(f(q))$$

$$\min_{q \in M(\pi g, \pi f)} u(g(q)) = \min_{q \in M^*(\pi g, \pi f)} u(g(q))$$

Hence, $M^*: \Pi \times \Pi \to \mathcal{M}$ is symmetric and satisfies (40).

If $\succeq$ is transitive, then by Lemma 12, $M(\pi, \tau) = M(\pi, \pi)$ for all $\pi, \tau \in \Pi$.

Suppose that $\succeq$ is both monotonic and transitive. Let $M^* = M(\pi^*, \pi^*)$ where $\pi^* = \{\{\omega\} : \omega \in \Omega\}$ is the finest partition of the state space $\Omega$. Take any $\pi, \tau \in \Pi$.

By monotonicity,

$$M(\pi, \pi^*) = M_{\pi^*}(\pi, \pi^*) \supset M_{\pi^*}(\pi^*, \pi) = M(\pi^*, \pi)$$

and $M_{\pi^*}(\pi^*, \pi) \supset M_{\pi^*}(\pi, \pi^*)$. Thus $M_{\pi^*}(\pi, \pi) = M_{\pi^*}(\pi^*, \pi)$.

By transitivity,

$$M_{\pi^*}(\pi, \tau) = M_{\pi}(\pi, \tau) = M_{\pi}(\pi, \pi^*) = M_{\pi^*}(\pi^*, \pi) = M_{\pi^*}(\pi^*, \pi^*) = M_{\pi^*}.$$ 

Thus (40) holds if $M^*(\pi, \tau) = M^*$ for all $\pi, \tau$.

**Theorem 3**

Let $\succeq$ satisfy Axioms 1-4 and PB. Then $\succeq$ is represented by (38). For all $\pi, \tau \in \Pi$, select $p(\pi, \tau) \in M(\pi, \tau)$. By Lemma 12, PB implies that $M_\pi(\pi, \tau) = \{p_\pi(\pi, \tau)\}$.

Thus

$$V(f, \pi, \tau) = \min_{q \in M(\pi f, \pi g)} u(f(q)) = u(f(p(\pi, \tau))).$$

Thus (14) holds.

Uniqueness of $p_\pi(\pi, \tau)$ follows from the uniqueness of $M_\pi(\pi, \tau)$.

If $\succeq$ is monotonic, select $p(\pi, \tau) = p(\tau, \pi)$ from the set $M^*(\pi, \tau)$ defined in (41).

If $\succeq$ is transitive, select $p(\pi, \tau) = p(\pi, \pi)$ from $M(\pi, \pi)$.

If $\succeq$ is monotonic and transitive, then the set $M^* = M(\pi^*, \pi^*)$ is a singleton by Lemma 12.
Theorem 4
Let $\succeq$ satisfy CPT and Axioms 2–5. Then $\succeq$ is represented by (38). By Lemma 12, $M(\pi, \tau) = M(\pi, \pi \lor \tau)$ for all $\pi, \tau \in \Pi$. Suppose that $\succeq$ is monotonic. Then for all $\pi, \tau$,

$$M_\pi(\pi, \pi \lor \tau) \subset M_\pi(\pi \lor \tau, \pi) = M_\pi(\pi \lor \tau, \pi \lor \tau)$$

$$M_{\pi \lor \tau}(\pi \lor \tau, \pi \lor \tau) = M_{\pi \lor \tau}(\pi \lor \tau, \pi) \subset M_{\pi \lor \tau}(\pi, \pi \lor \tau).$$

As $\pi \lor \tau$ is finer than $\pi$, then $M_\pi(\pi, \tau) = M_\pi(\pi, \pi \lor \tau) = M_\pi(\pi \lor \tau, \pi \lor \tau)$. Thus (40) holds with $M^*(\pi, \tau) = M(\pi \lor \tau, \pi \lor \tau)$ for all $\pi, \tau$.

Suppose that $\succeq$ satisfies PB. Select $p(\pi, \tau) = p(\pi \lor \tau, \pi \lor \tau)$ from $M(\pi, \pi \lor \tau)$. Suppose that $\succeq$ satisfies PB and monotonicity. Select $p(\pi, \tau) = p(\pi \lor \tau, \pi \lor \tau)$ from $M(\pi \lor \tau, \pi \lor \tau)$.

Theorem 5
Suppose that $\succeq$ is represented by (38).

Assume that $\theta_1 \in \Pi_\alpha$ is resolute. Take any $\pi, \tau \in \Pi$.

Let $L = \mathbb{R}^\Omega$ be the linear space of all vectors $a : \Omega \to \mathbb{R}$. Let $L_1$ be the set of all $\theta_1$-measurable vectors. Let $L_2$ be the set of all $\pi$-measurable vectors. Both $L_1$ and $L_2$ are linear subspaces. Constant vectors belong to the overlap $L_1 \cap L_2$.

Let $e \in L$ be the constant vector such that $e_\omega = 1$ for all $\omega \in \Omega$. Let $P = M(\theta_1, \{\Omega\})$ and $Q = M(\pi, \tau)$. For all $a \in L$, let

$$U_1(a) = \min_{q \in P} a \cdot q$$

$$U_2(a) = \min_{q \in Q} a \cdot q.$$ 

Both $U_1$ and $U_2$ are concave and satisfy

$$U_i(\alpha a + \beta e) = \alpha U_i(a) + \beta$$

for all $a \in L$, $\alpha > 0$, and $\beta \in \mathbb{R}$.

I claim that for all $a \in L_1$,

$$U_1(a) \succeq U_2(a). \quad (42)$$

Suppose that $U_1(b) < U_2(b)$ for some $b \in L_1$. As both $U_1$ and $U_2$ are continuous functions, then $b$ can be taken so that its generated partition is $\theta_1$. Rescale $b \in L_1$ so that $b_\omega \in [-1, 1]$ for all $\omega \in \Omega$. Take $\gamma \in [-1, 1]$ such that $U_1(b) < \gamma < U_2(b)$.

Take $f \in A_{\theta_1}$ and $x \in X$ such that $u(f(\omega)) = b_\omega$ for all $\omega \in \Omega$, and $u(x) = \gamma$. Note that $V(f, \theta_1, \{\Omega\}) = U_1(b)$ and $V(f, \pi, \tau) = U_2(b)$. By definition of $V$, there is $g \in A_x$ and $h \in A_\tau$ such that

$$\alpha f + (1 - \alpha) c_0 \succeq g \triangleright h \succeq \alpha x + (1 - \alpha) c_0.$$
As $\theta_1$ is resolute and $f \in \mathcal{A}_{\theta_1}$, then $x \succeq f$ implies

$$ h \geq \alpha x + (1 - \alpha)c_0 \geq \alpha f + (1 - \alpha)c_0 \geq g $$

and hence, $h \succeq g$. This contradiction implies that $f \succ x$. Yet $f \succ x$ contradicts $V(f, \theta_1, \{\Omega\}) < u(x)$. Thus claim (42) is true.

Similarly, for all $a \in L_2$,

$$ U_2(a) \geq U_1(a). \hspace{1cm} (43) $$

Suppose that $U_2(b) < U_1(b)$ for some $b \in L_2$. Then there is $f \in \mathcal{A}_x$ and $x, y \in X$ such that $V(f, \pi, \tau) < u(y) < u(x) < V(f, \theta_1, \{\Omega\})$. Take $h \in \mathcal{A}_x$ such that $x \gg h \gg y$. By definition of $V$, there is $g \in \mathcal{A}_{\theta_1}$ and $z \in X$ such that

$$ \alpha f + (1 - \alpha)c_0 \geq g \succ z \geq \alpha x + (1 - \alpha)c_0 \geq ah + (1 - \alpha)c_0. $$

As $\theta_1$ is resolute and $g \in \mathcal{A}_{\theta_1}$, then $\alpha f + (1 - \alpha)c_0 \geq ah + (1 - \alpha)c_0$. Thus $f \geq h$ and hence, $V(f, \pi, \tau) \geq V(h, \pi, \tau) \geq u(y)$. This contradiction implies claim (43).

By Hahn-Banach extension theorem (e.g. Aliprantis and Border [2, Theorem 5,40]), (42) implies that for each $q \in P$, there is a linear functional $l$ such that $l(a) = q \cdot a$ for all $a \in L_1$ and $l(a) \geq U_2(a)$ for all $a \in L$.

As $l$ is linear, then there is $r \in \mathbb{R}^\Omega$ such that $l(a) = r \cdot a$. As $l(a) \geq U_2(a) \geq 0$ for all positive vectors $a$, then $r \in \mathbb{R}^\Omega$. As $l(e) \geq U_2(e) = 1$ and $l(-e) \geq U_2(-e) = -1$, then $l(e) = 1$. Thus $r \in \Delta$. As $r \cdot a \geq U_2(a)$ for all $a \in L$, then $r \in Q$. As $r \cdot a = q \cdot a$ for all $a \in L_1$, then $r_{\theta_1} = q_{\theta_1}$. By Lemma 12, $r \in P$. Thus $(P \cap Q)_{\theta_1} = P_{\theta_1}$.

Similarly, (43) implies that $(P \cap Q)_x = Q_x$. By Lemma 12, the BPMEU model (40) still holds for

$$ M^*(\pi, \tau) = M(\pi, \tau) \cap M(\theta_1, \{\Omega\}). $$

Moreover, the projection of this set to $\theta_1$ equals $M_{\theta_1}(\theta_1, \{\Omega\})$ for all $\pi, \tau$.

Take any other resolute partition $\theta_2 \in \Pi_x$. Repeat the above argument for $P = M(\theta_2, \{\Omega\})$ and $Q = M(\pi, \tau) \cap M(\theta_1, \{\Omega\})$. Then the BPMEU model (40) still holds for

$$ M^*(\pi, \tau) = M(\pi, \tau) \cap M(\theta_1, \{\Omega\}) \cap M(\theta_2, \{\Omega\}). $$

Moreover, the projection of this set to $\theta_2$ equals $M_{\theta_2}(\theta_2, \{\Omega\})$.

As there are finitely many partitions, this process will deliver a non-empty set $M^*(\pi, \tau)$ that satisfies (40) and equals $M_{\theta}(\theta, \{\Omega\})$ for all resolute partitions $\theta$.

Repeat for all $\pi, \tau$. For any separable partition $\theta \in \Pi_x$, Independence over $\theta$-measurable acts implies that $M_{\theta}(\theta, \{\Omega\})$ must be a singleton.

If $\succeq$ satisfies PB, then by Lemma 12, the projection of $p(\pi, \tau)$ to $\theta$ is a singleton. Hence all resolute partitions are separable. Theorem 5 follows.
Theorem 7

Suppose that \( \succeq \) is represented by
\[
f \succeq g \iff \min_{q \in M(\pi_f \lor g)} u(f(q)) \geq \min_{q \in M(\pi_f \lor g)} u(g(q))
\]
for some \( M : \Pi \to \mathcal{M} \).

Assume comparative ignorance: for all \( x, y \in X, E \in \mathcal{T}, \) and \( g \in \mathcal{H}, \)
\[x \succ y \succeq g \Rightarrow xEy \succeq g.\]

For any event \( E \subset \Omega \) and partition \( \pi \in \Pi, \) let \( \pi \lor E = \pi \lor \{E, \neg E\} \) be the coarsest partition that contains both \( \pi \) and \( E.\)

I claim that for all \( \pi \in \Pi \) and separable events \( E \subset \Omega, \)
\[M_\pi(\pi \lor E) \supset M_\pi(\pi).\]

Let \( \pi = \{E_1, \ldots, E_n\}. \) Interpret each element \( q \in \Delta_\pi \) as a vector in \( \mathbb{R}^n. \) Suppose that there is \( p \in M_\pi(\pi) \) such that \( p \not\in M_\pi(\pi \lor E). \) As \( M_\pi(\pi) \) and \( M_\pi(\pi \lor E) \) are both closed and convex, then there is a vector \( a \in \mathbb{R}^n \) such that
\[
\min_{q \in M_\pi(\pi)} q \cdot a < \min_{q \in M_\pi(\pi \lor E)} q \cdot a.
\]

Wlog \( a \in [-1,1]^n \) and \( a \) generates \( \pi. \) Take \( g \in \mathcal{A}_\pi \) such that \( u(g(\omega)) = a_\omega \) for all \( \omega \in \Omega. \) Take \( x, y \in X \) such that
\[V(g, \pi, \{\Omega\}) = \min_{q \in M_\pi(\pi)} q \cdot a < u(y) < u(x) < \min_{q \in M_\pi(\pi \lor E)} q \cdot a = V(g, \pi \lor E, \{\Omega\}).\]

Then \( x \succ y \succeq g, \) but \( g \succeq xEy \succeq y. \) This contradiction implies that \( M_\pi(\pi \lor E) \supset M_\pi(\pi). \)

By induction, \( M_\pi(\pi \lor \theta) \supset M_\pi(\pi) \) for all \( \pi \in \Pi \) and separable partitions \( \theta \in \Pi_s. \)

References


