Combinatorial Models of Subjective States

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Abstract

I identify subjective state spaces for three structures—complete and transitive preferences, incomplete dominance relations, and choice functions—that are given on finite menus. These structures can be modelled together or separately. My results employ some combinatorial techniques and motivations. In particular, up to \( k \) subjective states can be derived from monotonic preferences over menus that have at most \( k \) elements, or from choices in menus that have at most \( k + 1 \) elements. Such applications require finding state spaces of minimal size, which can be done through standard combinatorial algorithms that partition a poset into a minimal number of chains.

For preferences, the additive representation of Kreps (1979) is relaxed to a weaker model called coherent aggregation. Coherent aggregation can be non-monotonic and hence, accommodate preferences for commitment. The case of a singleton state space delivers Gul and Pesendorfer’s (2005) model of changing tastes. For dominance relations, subjective states are identified through Pareto representations. Choice functions are rationalized via strict maximization of subjective states, which extends the model of Aizerman and Malishevski (1981).

1 Introduction

Kreps [25] uses subjective states to represent preferences \( \succeq \) over the domain \( \mathcal{M} \) of all menus—non-empty subsets \( A \) of a finite consumption domain \( Z \). The Krepsian aggregation model has the additive form

\[
U(A) = \sum_{\omega \in \Omega} \max_{z \in A} \omega(z),
\]

\[ (1) \]

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where \( \Omega \) is some collection of functions \( \omega : Z \to \mathbb{R} \). The decision maker (DM) as portrayed by (1) believes ex ante that her ex post choice in any menu \( A \) should maximize one of the functions \( \omega \in \Omega \). Her subjective state space \( S \) is identified as the collection of all rankings \( R \) that are represented by the functions \( \omega \in \Omega \) on the consumption domain \( Z \).

A well-known critique of the Krepsian model (1) is that the subjective state space \( S \) need not be unique. Dekel, Lipman, Rustichini [8] (henceforth, DLR) provide several examples of such non-uniqueness and discuss why it can be undesirable in applications. DLR show that if menus consist of lotteries, then preferences \( \succeq \) determine a unique subjective state space \( S \) within the class of expected utility rankings. However, DLR's identification of subjective states relies on probabilistic mixtures of menus of lotteries, the expected utility hypothesis, and local variations of utility functions. All of these features can be problematic in empirical settings.

In this paper, I identify subjective state spaces via a new strategy that should be more robust empirically than the methods of Kreps and DLR. The strategy relies on combinatorial techniques and finite menus. It does not guarantee uniqueness, but has several other benefits.

- Primitives are simplified. No lotteries or probabilities are imposed to identify subjective states. It can be sufficient to observe preferences only over small menus in \( M \). The combinatorial model identifies up to \( k \) subjective states from monotonic preferences \( \succeq \) over menus \( A \subset Z \) that have up to \( k \) elements. The number of such menus is polynomial, while the size of \( M \) is exponential. Moreover, sets of small size can have additional interpretations, such as couples in matching problems (e.g. Kojima, Pathak, and Roth [20]).

- Subjective state spaces can be derived for any complete and transitive preference \( \succeq \) that can violate monotonicity and submodularity assumed by Kreps. For example, the non-monotonic representation with a singleton state is the model of changing tastes that Gul and Pesendorfer [17, 18] propose to accommodate preferences for commitment.

- The structure of all spaces \( S \) that represent a given preference \( \succeq \) is explicitly characterized in terms of the family \( F \) of all focal menus derived endogenously from \( \succeq \). The state space \( S \) of minimal size can be found via standard combinatorial algorithms that partition \( F \) into a minimal number of chains.\(^1\)

- Besides preferences over menus, choice functions can be also used to identify subjective states (or rationales). Given a pair of conditions relaxing path independence, choice functions can be rationalized by a state space \( S \) that

\(^1\)By contrast, it is much harder to compute the minimal number of states in the additive representation (1). Kopylov and Zhao [24] provide a partial answer for two states, but even this special case requires solving a system of linear inequalities. The answer for three or more states is an open problem that appears very complex.
can be also constructed via combinatorial methods. The path independent model of Aizerman and Malishevski [2] is a special case where all states are total orders on $Z$.

- Up to $k$ subjective states can be found from choices in menus of size up to $k + 1$. Moreover, there is a broader algorithm to check whether a choice function $C$ given on an arbitrary subdomain $D \subset M$ can be rationalized by some subjective state space $S$.

Consider some motivational examples.

### 1.1 Examples

Let $k = 2$ and $Z = \{a, b, c\}$. With some abuse of notation, write singleton menus like \{a\} as $a$, and small menus like \{a, b\} as $ab$. Consider a preference $\succeq$

$$bc \succ ac \succ ab \sim a \succ c \succ b$$

over the domain $D$ of all menus in $Z$ that have at most two elements.

The combinatorial model of subjective states implies that $\succeq$ is represented by

$$U(A) = \phi(\max_{z \in A} \omega_1(z), \max_{z \in A} \omega_2(z)) \quad \text{for all } A \in D,$$

where the functions

$$\omega_1(b) = 2 > \omega_1(a) = 1 > \omega_1(c) = 0$$
$$\omega_2(c) = 2 > \omega_2(a) = 1 > \omega_2(b) = 0$$

reflect two subjective states and $\phi$ is a monotonic aggregator such that

$$\phi(2, 2) > \phi(1, 2) > \phi(2, 1) = \phi(1, 1) > \phi(0, 2) > \phi(2, 0).$$

The two subjective states $R_1$ and $R_2$ represented by $\omega_1$ and $\omega_2$ are determined uniquely in this example.

By contrast, preference (2) does not have any additive utility representation (1) because any monotonic extension of $\succeq$ to $M$ must satisfy $ab \sim a$ and $abc \succ ac$. Thus it violates the submodularity condition assumed by Kreps.

Consider another example. Let $Z = \{a, b, c, d\}$ and preference $\succeq$ be such that

$$ad \succ ac \succ bd \succ bc \succ cd \succ ab \succ b \succ c \succ a \succ d$$

over the domain $D$ of all menus that have at most two elements. The combinatorial model implies that $\succeq$ is represented by (3) with

$$\omega_1(a) = 3 > \omega_1(b) = 2 > \omega_1(c) = 1 > \omega_1(d) = 0$$
$$\omega_2(d) = 3 > \omega_2(c) = 2 > \omega_2(b) = 1 > \omega_2(a) = 0,$$
and a strictly monotonic aggregator $\phi$ such that

$$\phi(3, 3) > \phi(3, 2) > \phi(2, 3) > \phi(2, 2) > \phi(1, 3) >$$

$$> \phi(3, 1) > \phi(2, 1) > \phi(1, 2) > \phi(3, 0) > \phi(0, 3).$$

The two subjective states $R_1$ and $R_2$ represented by $\omega_1$ and $\omega_2$ are determined uniquely in this example. Moreover, representation (3) implies a unique extension of $\succeq$ to the entire $\mathcal{M}$:

$$Z \sim abd \sim acd \sim ad \succ abc \sim acd \sim bcd \sim bd \succ bc \succ cd \succ ab \succ b \succ c \succ a \succ d$$

where each menu is as good as its best two-element subset. This extension satisfies the Krepsian axioms, but it does not have any two-state additive representation

$$U(A) = \max_{z \in A} \omega_1^*(z) + \max_{z \in A} \omega_2^*(z) \quad \text{for all } A \in \mathcal{D}. \quad (5)$$

One needs at least three states to represent $\succeq$ by (1). Not only (5) fails to exist, establishing this fact is not easy (see Section 4.) Thus the Krepsian model (5) is less general and less convenient for modeling preferences like (2) and (4).

When $\succeq$ is given on the entire $\mathcal{M}$, the evaluation of larger menus can be affected by complexity, limited attention, and other behavioral phenomena that may have little or no effect on comparisons between small menus. Thus it can be natural to obtain a separate representation for preferences over small menus. The combinatorial model accommodates such applications, but the additive aggregation model can fail to do so, as illustrated above.

A related issue with the Krepsian model is its dependence on indifferences between nested menus. For example, any strict ranking $Z \succ Z \setminus \{a\}$ implies that there must be a subjective state where $a$ is the best element in the entire $Z$. By contrast, any indifference $Z \sim Z \setminus \{a\}$ implies that no such state should exist. However, distinguishing indifference between two nested menus $A \supseteq B$ can be difficult both because of limited attention (e.g. some agents may fail to notice the difference between $Z$ and $Z \cup \{a\}$) and because the larger menu $A$ can be chosen over $B$ through a simple heuristic principle—keep as much flexibility as possible—without any additional cognitive effort.

By contrast, subjective states that are derived from preferences over menus that have at most $k$ elements are unaffected by any comparisons between nested menus of size more than $k$. In fact, combinatorial models can be applied even when preferences are restricted to menus with precisely $k$ elements. In this case, subjective states can be derived independently of any choices between nested menus.

Similar applications can be obtained for choice functions. Suppose that $Z = \{a, b, c, d, e\}$, and choices $C(A)$ are observed in all menus $A$ that have up to three
elements. Let $C(z) = z$ for all $z \in Z$, and

\[
\begin{align*}
C(ab) &= ab & C(ac) &= a & C(ad) &= a & C(ae) &= ae & C(bc) &= b \\
C(bd) &= b & C(be) &= b & C(cd) &= c & C(ce) &= ce & C(de) &= de \\
C(abc) &= ab & C(abd) &= ab & C(abe) &= ab & C(acd) &= a & C(ace) &= ae \\
C(ade) &= ae & C(bcd) &= b & C(bce) &= b & C(bde) &= b & C(cde) &= ce.
\end{align*}
\]

The combinatorial methods imply that $C$ can be rationalized by two states $R_1$ and $R_2$ (rankings of $Z$) such that

$$C(A) = \max_A R_1 \cup \max_A R_2$$

for all $A \in D$, \hspace{1cm} (6)

where each set $\max_A R$ consists of all strict maximizers of the ranking $R$ in the menu $A$. In this example, both $R_1$ and $R_2$ can be found as total orders with asymmetric components $P_1$ and $P_2$ such that $aP_1bP_1cP_1dP_1e$ and $bP_2eP_2aP_2cP_2d$. Moreover, the pair of total orders $R_1$ and $R_2$ that rationalizes $C$ is unique.\(^2\)

When $R_1$ and $R_2$ are total orders and $D = M$, representation (6) is a special case of Aizerman and Malishevski’s [2] multi-preference model for path independent choice functions. However, their results do not deliver (6) because they do not identify the minimal number of subjective states that rationalize $C$ on $M$. Combinatorial techniques are required to make this step.

More generally, I provide a broad algorithm to check whether a choice function $C$ given on an arbitrary subdomain $D \subset M$ can be rationalized by some subjective state space $S$.

1.2 Utility Functions and Other Representations

Subjective state spaces $S = \{R_1, \ldots, R_k\}$ are modeled as sets of rankings $R_i$ of $Z$. Each $R_i$ can be represented by some function $\omega_i : Z \rightarrow \mathbb{R}$. Subjective state spaces are derived for three distinct, but related structures over all menus $A, B \in M$:

- complete and transitive preferences $\succeq$ are represented by

$$U(A) = \phi(\max_{z \in A} \omega_1(z), \ldots, \max_{z \in A} \omega_k(z))$$ \hspace{1cm} (7)

where the aggregator $\phi$ satisfies a mild property called coherence, but need not be additive or monotonic,

- incomplete dominance relations $\succeq$ over $M$ have Pareto representations

$$A \succeq B \iff \max_{z \in A} \omega_i(A) \geq \max_{z \in B} \omega_i(B) \quad \text{for all } i = 1, \ldots, k \hspace{1cm} (8)$$

\(^2\)In general, uniqueness is not guaranteed even if only two states are allowed.
• choice functions $C: \mathcal{M} \rightarrow \mathcal{M}$ are rationalized by

$$C(A) = \{z \in A \text{ is a strict maximizer in } A \text{ for some } R_i \in S\} \quad (9)$$

where $C(A)$ may be empty for some menus.\textsuperscript{3}

If representations (7)—(9) hold with the same $S$, then both $\succeq$ and $C$ can be derived from $\succeq$ or from each other endogenously.

Theorems 3 and Theorem 11 below establish that a subjective state space $\mathcal{S}$ satisfies either one of representations (7)—(9) if and only if the set inclusions

$$\mathcal{F} \subset \mathcal{L} \subset \pi(\mathcal{F}) \quad (10)$$

hold where

- $\mathcal{L}$ is the family of all lower contour sets of the rankings $R_1, \ldots, R_k$,

- $\mathcal{F}$ is the family of all focal menus that are defined endogenously in terms of $\succeq$ (or $\succeq$, or $C$),

- $\pi(\mathcal{F})$ is the family of all intersections of focal menus.

In other words, any focal menu must be a lower contour set in some subjective state, and any lower contour set in any state must be an intersection of some focal menus. Condition (10) is the main engine for various algorithms developed in this paper. Most importantly, it identifies the minimal size of $\mathcal{S}$ as the width of the focus $\mathcal{F}$ with respect to the set inclusion.

1.3 Related Literature

Subjective states in the finite menu framework have been identified together with maxmin aggregation in Natenzon [27] and non-monotonic additive aggregation in Gorno [16]. Pareto representations (8) are obtained in the Krepsian model by Nehring and Puppe [28] and in the general framework, by Danilov, Koshevoy, and Savaglio [7]. A distinct benchmarking model for incomplete preferences over finite menus is proposed by Chambers and Miller [6]. The set of essential elements defined by Puppe [30] translates preferences $\succeq$ in the Krepsian model into choice functions. None of these models discuss the minimal size of subjective state spaces, coherent aggregation, and combinatorial applications of subjective state spaces.

Starting from DLR, there is a large literature where subjective state spaces are derived in an essentially unique way from preferences over compact subsets $A$ of a lottery space $Z$. This literature includes the model of costly self-control\textsuperscript{4} in

\textsuperscript{3}$C(A)$ is guaranteed to be non-empty if all $R_i$ are total orders on $Z$.

\textsuperscript{4}This model is obtained independently of DLR’s results.
Gul and Pesendorfer [17]), costly contemplation in Ergin and Sarver [11], random choice in Ahn and Sarver [1], and many other applications.

One of DLR’s utility representations is

\[
U(A) = \sum_{\omega \in \Omega_+} \max_{z \in A} \omega(z) - \sum_{\omega \in \Omega_-} \max_{z \in A} \omega(z) \tag{11}
\]

where subjective states in \(\Omega_+\) and \(\Omega_-\) are expected utility functions \(\omega : Z \to \mathbb{R}\). Note that the minimal numbers of both positive and negative components in (11) can be expressed explicitly in terms of preference (see Kopylov [22]), but this procedure is radically different from its counterpart in the finite menu framework.

Finally, representation (8) adds to the large literature that derives Pareto representations for incomplete preferences. Such models have been studied on general domains (Evren and Ok [12]) and over more specific alternatives, such as lotteries (Dubra, Maccheroni, Ok [10], Cerreia-Vioglio, Dillinberger, and Ortoleva [5]) and uncertain prospects (Bewley [4], Ghirardato, Maccheroni, and Marinacci [14], Gilboa, Maccheroni, Marinacci and Schmeidler [15], Kopylov [21] and others). These representations can accommodate indecisiveness, group decisions, ambiguity aversion, certainty effects, objective rationality, model uncertainty, choice deferral and other phenomena.

2 Preliminaries

Take a finite consumption set \(Z = \{a, b, c, x, \ldots\}\). Let \(\mathcal{M} = \{A, B, C, \ldots\}\) be the set of all menus—non-empty subsets of \(Z\). With some abuse of notation, write singletons \(\{x\}\) as \(x\) and finite menus like \(\{a, b, c\}\) as \(abc\).

Interpret each menu \(A \in \mathcal{M}\) as an action that, if taken ex ante, makes the set \(A \subset Z\) feasible ex post. Consider a decision maker (DM) who has a weak preference \(\succeq\) over menus. Write its asymmetric and symmetric parts as \(\succ\) and \(\sim\) respectively. It is assumed throughout that \(\succeq\) is a weak order, that is, complete and transitive.

The induced dominance of the preference \(\succeq\) is a binary relation \(\succeq\) such that for all menus \(A, B \in \mathcal{M}\),

\[
A \succeq B \iff A \cup D \sim A \cup B \cup D \quad \text{for all } D \in \mathcal{M}. \tag{12}
\]

Say also that \(\succeq\) is induced by \(\succeq\). The asymmetric and symmetric parts of \(\succeq\) are denoted as \(\gg\) and \(\cong\) respectively.

By definition, \(\succeq\) satisfies the following properties for all \(A, B, C \in \mathcal{M}\).

**Axiom 1** (Monotonicity). \(A \cup B \succeq A\).

**Axiom 2** (Transitivity). If \(A \succeq B \succeq C\), then \(A \succeq C\).
**Axiom 3** (Additivity). If $A \succeq B$, then $A \cup C \succeq B \cup C$.

Take any $A, B, C \in \mathcal{M}$. Monotonicity is obvious. If $A \succeq B \succeq C$, then $A \succeq C$ because for all $D \in \mathcal{M}$,

$$A \cup D \sim A \cup B \cup D \sim A \cup B \cup C \cup D \sim A \cup C \cup D.$$ 

Thus $\succeq$ is transitive. If $A \succeq B$, then $A \cup C \succeq B \cup C$ because for all $D \in \mathcal{M}$,

$$A \cup C \cup D \sim A \cup B \cup (C \cup D) = (A \cup C) \cup (B \cup C) \cup D.$$ 

Thus $\succeq$ is additive.

For any utility function $U : \mathcal{M} \to \mathbb{R}$, its induced dominance $\succeq$ equals the induced dominance of the weak order $\succeq$ represented by $U$. Thus, for all $A, B \in \mathcal{M}$,

$$A \succeq B \iff U(A \cup D) = U(A \cup B \cup D) \quad \text{for all } D \in \mathcal{M}.$$ 

Note that the induced dominance $\succeq$ satisfies Axioms 1–3 even if $\succeq$ is a partial order (i.e. reflexive, transitive, but not necessarily complete). However, any utility representation $U : \mathcal{M} \to \mathbb{R}$ implies that $\succeq$ is complete.

### 2.1 Subjective states and state spaces

Let $\mathcal{R} = \{R, \ldots\}$ the set of all complete and transitive orders on $Z$. For any $R \in \mathcal{R}$, its asymmetric and symmetric parts are written as $P$ and $I$ respectively.

Subjective states and state spaces are derived as rankings $R \in \mathcal{R}$ and sets $S \subset \mathcal{R}$ respectively.

Let $\mathbb{R}^Z$ be the set of all functions $\omega : Z \to \mathbb{R}$. For any menu $A \in \mathcal{M}$ and $\omega \in \mathbb{R}^Z$, write the maximal value of $\omega$ in $A$ as

$$\omega(A) = \max_{z \in A} \omega(z).$$

Any ranking $R \in \mathcal{R}$ has a natural extension from $Z$ to $\mathcal{M}$: for all $A, B \in \mathcal{M}$,

$$ARB \iff \text{there is } x \in A \text{ such that } xRy \text{ for all } y \in B.$$ 

Take any $\omega \in \mathbb{R}^Z$ that represents $R$ on $Z$. Then for all menus $A, B \in \mathcal{M}$,

$$ARB \iff \omega(A) \geq \omega(B).$$

Thus the natural extension of any $R \in \mathcal{R}$ is complete and transitive on $\mathcal{M}$.

For any $R \in \mathcal{R}$, its label $L(R)$ is the family of lower contour sets

$$L(y, R) = \{z \in Z : yRz\}$$

for all $y \in Z$. All sets $L(y, R)$ are non-empty and hence, menus. Any $R \in \mathcal{R}$ can be uniquely reconstructed from its label $L(R)$. 

8
A family $D \subset M$ is called a chain if for all $A, B \in D$, either $A \supset B$ or $B \supset A$. For any chain $D$, define the function $\omega_D \in \mathbb{R}^Z$ such that for all $z \in Z$, 

$$\omega_D(z) = |\{A \in D : z \notin A\}|.$$ 

Note that $\omega_D(z)$ is the number of menus in $D$ that do not contain $z$.

**Proposition 1.** $D \subset M$ is a chain if and only if there is $R \in \mathcal{R}$ such that $D \cup \{Z\} = \mathcal{L}(R)$.

Moreover, such $R$ is unique and represented by $\omega_D$.

**Proof.** Note that for any $R \in \mathcal{R}$ and $x, y \in Z$, $xRy$ if and only if for all $A \in \mathcal{L}(R)$, $x \in A$ implies $y \in A$. Thus $R$ is uniquely determined by its label $\mathcal{L}(R)$.

Take any chain $D$ and let $R$ be represented by $\omega_D$. All menus in the chain $D \cup \{Z\}$ can be ordered so that $D_1 \subset D_2 \subset \ldots D_{k-1} \subset D_k = Z$.

Let $D_0 = \emptyset$. For any $i = 1, \ldots, k$, take $z_i \in D_i \setminus D_{i-1}$. Then $L(z_i, R) = D_i$. Moreover, for any $z \in Z$, let $j$ be the smallest index such that $z \in D_j$. Then $L(z, R) = D_j$. Thus $D \cup \{Z\} = \mathcal{L}(R)$.

For example, for any menu $A \in \mathcal{M}$, there is a unique $R_A \in \mathcal{R}$ such that $\mathcal{L}(R_A) = \{A, Z\}$.

This ranking is called Boolean because it is represented by the Boolean function $\omega_A : Z \to \{0, 1\}$ such that $\omega_A(z) = 1$ for all $z \notin A$, and $\omega_A(z) = 0$ for all $z \in A$.

For any space $S \subset \mathcal{R}$, its label

$$\mathcal{L}(S) = \bigcup_{R \in S} \mathcal{L}(R)$$

is the family of lower contour sets $L(y, R)$ for all $y \in Z$ and $R \in S$.

For any family $D \subset \mathcal{M}$, let $\mathcal{B}(D) = \{R_A : A \in D\}$ consist of all Boolean rankings $R_A$ for $A \in \mathcal{D}$. Then

$$\mathcal{L}(\mathcal{B}(D)) = D \cup \{Z\}.$$ (13)

Thus any family of menus that contains $Z$ is the label of some state space.

Unlike single states $R \in \mathcal{R}$, state spaces are not determined uniquely by their labels. For example, if $Z = \{a, b, c\}$, then

$$\mathcal{L}(\{R_a, R_{ab}\}) = \{a, ab, Z\} = \mathcal{L}(\{R\})$$

where $R$ is such that $cPbPa$. The ranking $R$ is represented by $\omega_D$ for $D = \{a, ab, Z\}$ because $\omega_D(c) = 2$, $\omega_D(b) = 1$, and $\omega_D(a) = 0$. 

2.2 Krepsian Aggregation Model

Take any weak order $\succeq$ on $\mathcal{M}$.

Say that $\succeq$ is monotonic if $A \cup B \succeq A$ for all $A, B \in \mathcal{M}$.

Say that $\succeq$ is submodular if for all $A, B, D \in \mathcal{M},$

$$A \sim A \cup B \Rightarrow A \cup D \sim A \cup B \cup D.$$  

These axioms can be interpreted in terms of the induced dominance $\geq$. If $\succeq$ is monotonic, then $A \geq B$ implies $A \sim A \cup B \succeq B$ and hence, $A \succeq B$. Conversely, if $\geq$ is a subrelation of $\succeq$, then $\succeq$ is monotonic because $\geq$ is. Thus $\succeq$ is monotonic if and only if $\geq$ is a subrelation of $\succeq$.

Submodularity holds if and only if for all $A, B \in \mathcal{M},$

$$A \geq B \iff A \sim A \cup B.$$  

(14)

Kreps [25] uses (14) as a definition of the auxiliary relation $\preceq$ in his proofs. Given submodularity, (14) is equivalent to (12).

The celebrated theorem of Kreps [25] asserts that $\succeq$ is complete, transitive, monotonic, and submodular if and only if $\succeq$ is represented by

$$U_\Omega(A) = \sum_{\omega \in \Omega} \omega(A)$$  

(15)

for some space $\Omega \subset \mathbb{R}^Z$. The DM as portrayed by (15) believes ex ante that her ex post choice will maximize some function $\omega \in \Omega$ and puts equal weight on each maximal value $\omega(A)$. The equality of weights is wlog: if arbitrary positive weights are assigned to $\omega(A)$, then (15) still holds for properly rescaled $\omega$.

In this interpretation, the subjective state space

$$\mathcal{R}_\Omega = \{R \in \mathcal{R} : R \text{ is represented by some } \omega \in \Omega\}$$

consists of all ex post rankings that the DM views as possible ex ante. The induced dominance $\geq$ provides another interpretation for $\mathcal{R}_\Omega$.

**Proposition 2.** For any set $\Omega \subset \mathbb{R}^Z$, the function $U_\Omega$ has the induced dominance $\geq$ such that for all $A, B \in \mathcal{M},$

$$A \geq B \iff ARB \text{ for all } R \in \mathcal{R}_\Omega.$$  

(16)

**Proof.** Suppose that $\succeq$ is represented by $U_\Omega$, and $\geq$ is induced by $\succeq$. If $\omega(A) \geq \omega(B)$ for all $\omega \in \Omega$, then $A \geq B$ because for all $D \in \mathcal{M}$, $\omega(A \cup D) = \omega(A \cup B \cup D)$ and hence by (15), $A \cup D \sim A \cup B \cup D$. On the other hand, if $\omega(B) > \omega(A)$ for some state $\omega \in \Omega$, then by (15), $A \cup B \sim A \cup B \succeq A = A \cup A$ and hence, $A \geq B$ does not hold. Thus for all $A, B \in \mathcal{M},$

$$A \geq B \iff \omega(A) \geq \omega(B) \text{ for all } \omega \in \Omega$$

which implies (16). \qed
Thus the standard Pareto criterion (16) with the subjective state space $R_{Ω}$ represents the induced dominance $≧$ in the Krepsian model (15). This observation is due to Nehring and Puppe [28]. In this paper, I apply Pareto representations (16) more broadly.

3 Subjective States via Dominance Relations

Take any binary relation $≧$ over $M$. Let $≫$ and $≃$ be its strict and symmetric components respectively. In particular, $≧$ can be the induced dominance for some preference $⪰$, but other specifications are possible.

Say that $≧$ is a dominance relation (or dominance for short) if $≧$ is transitive, monotonic, and additive.

Say that $≧$ is represented by a state space $S ⊂ R$ if for all $A, B ∈ M$,

$$A ≩ B ⇔ ARB$$

for all $R ∈ S$ (17) where the notation $ARB$ refers to the natural extension of $R$ from $Z$ to $M$.

Say that a menu $F ∈ M$ is focal for $y ∈ Z$ if

$$F ∪ x ≩ F ∪ y ≫ F$$

for all $x ∈ F$.

Say that $F ∈ M$ is focal if it is focal for some $y ∈ Z$.

The focus $F(≧)$ is the family of all focal menus of the dominance $≧$. When the relation $≧$ is clear from the context, $F(≧)$ is written as $F$ for brevity.

For any family $D ⊂ M$, let

$$π(D) = \left\{ A ∈ M : A = \bigcap_{D ∈ D : A ⊂ D} D \right\}$$

be the collection of all non-empty intersections of menus from the family $D$. Obviously, $D ⊂ π(D)$. By convention, $Z$ is the intersection of the empty family $∅ ⊂ M$. Thus $Z ∈ π(D)$ for any $D$.

My main representation result for dominance relations is

**Theorem 3.** A dominance $≧$ is represented by a state space $S ⊂ R$ if and only if

$$F ⊂ L(S) ⊂ π(F).$$

(18)

All theorems are shown in the appendix; other claims are proven immediately. This result characterizes all state spaces $S$ that represent a dominance $≧$ via the set inclusions (18). The proof is in the appendix.

Theorem 3 has many implications that are discussed next.
**Corollary 4.** $\succeq$ is a dominance relation if and only if $\succeq$ is represented by some state space $S \subset \mathcal{R}$.

In particular, any dominance $\succeq$ is represented by $S = B(\mathcal{F})$.

*Proof.* For any dominance $\succeq$, take $S = B(\mathcal{F}) = \{R_F : F \text{ is a focal menu}\}$. By (13), $L(S) = \mathcal{F} \cup \{Z\}$. By Theorem 3, $\succeq$ is represented by $S$. Conversely, if $\succeq$ is represented by some $S \subset \mathcal{R}$, then it is transitive, monotonic, and additive. \( \square \)

Corollary 4 is obtained by Danilov et al. [7] with an equivalent definition of dominance relations. Theorem 3 delivers a more constructive version where the state space $S$ is identified as the collection of Boolean states $B(\mathcal{F})$.

Corollary 4 implies further that for any two dominance relations $\succeq$ and $\succeq'$,

\[ \succeq = \succeq' \iff F(\succeq) = F(\succeq'). \]

Indeed, if $F(\succeq) = F(\succeq')$, then $\succeq$ and $\succeq'$ are represented by the same Boolean state space $B(\mathcal{F}(\succeq))$.

A function $U : \mathcal{M} \to \mathbb{R}$ is called Krepsian if it has the form $U = U_\Omega$ for some $\Omega \subset \mathbb{R}^Z$ or equivalently, $U$ represents a monotonic and submodular preference $\succeq$.

**Corollary 5.** $\succeq$ is a dominance relation if and only if $\succeq$ is induced by a Krepsian utility function $U$.

*Proof.* Take a dominance $\succeq$, and for all $A \in \mathcal{M}$, let

\[ U^*(A) = \sum_{F \in \mathcal{F}} \omega_F(A) = |\{F \in \mathcal{F} : A \setminus F \neq \emptyset\}|. \tag{19} \]

Note that $U^*(A)$ is the number of focal menus that do not contain $A$ as a subset. The function $U^*$ is Krepsian because $U^* = U_\Omega$ for $\Omega = B(\mathcal{F})$. Let $\succeq^*$ be the induced dominance of $U^*$. By Proposition 2, $\succeq^*$ is represented by $S = B(\mathcal{F})$. By Corollary 4, $\succeq$ is represented by the same $S = B(\mathcal{F})$. Thus $\succeq = \succeq^*$. \( \square \)

In applications, it may be desirable to impose some a priori conditions on the hypothetical subjective states. For example, if $Z$ consists of consumption bundles or uncertain prospects, then there are some natural monotonicity conditions that all states should satisfy. Suppose that such a priori considerations are given as a primitive partial order $\succeq_Z$ on $Z$. Assume that for all $x, y \in Z$,

\[ x \succeq_Z y \implies x \succeq y. \]

Then all subjective states $R \in S$ should satisfy $xRy$. Indeed, if $yPx$ for some $R \in S$, then $x \succeq y$ does not hold because $S$ represents $\succeq$. 

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### 3.1 Minimal State Spaces

Theorem 3 can be used to represent any dominance relation \( \geq \) via the smallest possible number of subjective states, which are usually not Boolean.

A state space \( \mathcal{S} \subset \mathcal{R} \) is called *minimal* for the dominance \( \geq \) if \( \mathcal{S} \) represents \( \geq \), and \( |\mathcal{S}| \leq |\mathcal{S}'| \) for any other \( \mathcal{S}' \subset \mathcal{R} \) that represents \( \geq \).

For any family \( \mathcal{D} \subset \mathcal{M} \), let the *width* \( W(\mathcal{D}) \) be the smallest number of chains that partition \( \mathcal{D} \). Formally, there is a partition

\[
\mathcal{D} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_{W(\mathcal{D})}
\]

of \( \mathcal{D} \) into \( W(\mathcal{D}) \) disjoint chains \( \mathcal{D}_1, \ldots, \mathcal{D}_{W(\mathcal{D})} \), and any partition of \( \mathcal{D} \) into \( k \) disjoint chains \( \mathcal{D}_1', \ldots, \mathcal{D}_k' \) must have \( k \geq W(\mathcal{D}) \) components. By convention, \( W(\emptyset) = 0 \) for empty \( \mathcal{D} \).

**Corollary 6.** For any state space \( \mathcal{S} \subset \mathcal{R} \) and dominance \( \geq \), the following statements are equivalent:

1. \( \mathcal{S} \) is minimal for \( \geq \),
2. \( |\mathcal{S}| = W(\mathcal{F}) \) and \( \mathcal{F} \subset \mathcal{L}(\mathcal{S}) \subset \pi(\mathcal{F}) \).

The proof is constructive.

**Proof.** Suppose that \( \geq \) is represented by \( k \) states \( \mathcal{S} = \{R_1, \ldots, R_k\} \subset \mathcal{R} \). By Theorem 3, \( \mathcal{F} \subset \mathcal{L}(\mathcal{S}) \) and hence, the focus \( \mathcal{F} \) is covered by \( k \) chains \( \mathcal{L}(R_1), \ldots, \mathcal{L}(R_k) \). Any subfamily of a chain is also a chain. Thus the focus \( \mathcal{F} \) can be partitioned into \( k \) chains \( \mathcal{D}_i \subset \mathcal{L}(R_i) \) for \( i = 1, \ldots, k \). Thus \( k \geq W(\mathcal{F}) \).

On the other hand, let \( k = W(\mathcal{F}) \), and partition the focus \( \mathcal{F} \) into \( k \) chains \( \mathcal{D}_1, \ldots, \mathcal{D}_k \). Take \( R_1, \ldots, R_k \) such that \( \mathcal{L}(R_i) = \mathcal{D}_i \cup \{Z\} \). Then

\[
\mathcal{F} \subset \bigcup_{i=1}^{k} \mathcal{L}(R_i) = \mathcal{F} \cup \{Z\} \subset \pi(\mathcal{F}).
\]

By Theorem 3, \( \geq \) is represented by \( \mathcal{S} = \{R_1, \ldots, R_k\} \), which is minimal. \( \Box \)

One of the steps in the above construction is partitioning \( \mathcal{F} \) into the minimal number of chains \( \mathcal{D}_1, \ldots, \mathcal{D}_k \). This is a well-known problem in combinatorics and computer science. Hopcroft and Karp [19] formulate an algorithm that finds the width and a minimal partition of any partial order (e.g. a family of menus) of size \( n \) in \( O(n^{5/2}) \) time. Felsner, Raghavan, and Spinrad [13] provide faster algorithms that recognize whether the width of a given partial order does not exceed \( k \). For \( k = 1, 2, 3 \) the recognition can be done in linear time.

Another useful result from combinatorics is Dilworth’s [9] Theorem that provides an equivalent definition of the width in terms of antichains.
A family $A \subset M$ is called an antichain if for all $A, B \in A$, $A \supset B$ implies $A = B$. Dilworth’s Theorem asserts that the width of any family $D \subset M$ is the largest size $|A|$ among all antichains $A \subset D$ that belong to $D$. In other words, $|A| = W(D)$ for some antichain $A \subset D$, and $|A| \leq W(D)$ for all antichains $A \subset D$. This equivalence implies

**Corollary 7.** For any $k = 1, 2, \ldots$ and any dominance $\succeq$, the following statements are equivalent:

(i) there are $R_1, \ldots, R_k \in R$ such that for all $A, B \in M$,

$$A \succeq B \iff AR_iB \text{ for all } i = 1, \ldots, k$$

(ii) for any $k + 1$ focal menus $F_1, \ldots, F_k, F_{k+1} \in \mathcal{F}$, there are $i, j \in \{1, \ldots, k+1\}$ such that $i \neq j$ and $F_i \subset F_j$.

Thus one can reject a $k$-state representation (20) for $\succeq$ by presenting $k + 1$ focal menus that do not have any nested pairs.

### 3.2 Total State Spaces

Say that $R \in \mathcal{R}$ is total if for any $x, y \in Z$, the indifference $xIy$ implies $x \neq y$.

Let $\mathcal{T} \subset \mathcal{R}$ be the class of all total orders on $Z$. A subjective state space $S$ is called total if $S \subset \mathcal{T}$.

Say that a chain $D \subset M$ is total if the chain $D \cup \{Z\}$ contains $|Z|$ menus. By Proposition 1, a chain $D$ is total if and only if $D = \mathcal{L}(R)$ for some total order $R$.

In some applications, it can be convenient to model subjective states as total orders. Say that $\succeq$ is weakly total if for all menus $A, B \in M$,

$$A \succeq B \Rightarrow A \cap B \succeq B.$$  

(21)

Note that $A \cap B \succeq B$ does not hold if $A \cap B$ is empty. Thus weakly total $\succeq$ prohibits the equivalence $A \succeq B$ for disjoint $A$ and $B$. For singleton menus $A$ and $B$, weak totality is equivalent to the standard totality for $\succeq$ because $A \cap B \neq \emptyset$ implies $A = B$.

**Theorem 8.** For any dominance $\succeq$, the following statements are equivalent,

(i) $\succeq$ is weakly total,

(ii) $\succeq$ is represented by a total state space $S \subset \mathcal{T}$,

(iii) $\succeq$ is represented by a total state space $S \subset \mathcal{T}$ that is also minimal,

(iv) for any chain $D \subset \pi(\mathcal{F})$, there is a total chain $D^* \subset \pi(\mathcal{F})$ such that $D \subset D^*$. 

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This result asserts that weak totality characterizes all dominance relations that can be represented by some total state space $S \subset T$. Whenever such a representation exists, it can be constructed to be minimal so that $|S| = W(F)$.

Next I discuss algorithms and examples for finding focal menus and various state spaces.

### 3.3 Algorithms and Examples

The first step in the identification of subjective states is the construction of focal menus. This construction can start from the dominance $\succeq$ or directly from other primitives that determine $\succeq$.

For any weak order $\preceq$, its focus $F(\preceq)$ equals the focus $F(\succeq)$ of the dominance $\succeq$ induced by $\preceq$.

For any state space $S \subset R$, its focus $F(S)$ equals the focus $F(\succeq)$ of the dominance $\succeq$ represented by $S$.

For any $R \in R$ and $y \in Z$, define the strict lower contour set

$$L'(y, R) = \{z \in Z : yPz\}.$$  

Say that a menu $A \in D$ is maximal in a family $D \subset 2^Z$ if for all $B \in D$, $A \subset B$ implies $A = B$.

Take any dominance $\succeq$, weak order $\preceq$, and state space $S \subset R$ such that $\succeq$ is induced by $\preceq$ and represented by $S$.

**Theorem 9.** For any menu $F \in M$ and element $y \in Z$, the following statements are equivalent:

1. $F$ is focal for $y$,
2. $F$ is maximal in the family $X(y, \succeq) = \{A \in M : A \cup y \succ A\}$,
3. $F$ is maximal in the family $X(y, \preceq) = \{A \in M : A \cup y \nsucceq A\}$,
4. $F$ is maximal in the family $X(y, S) = \{L'(y, R) : R \in S\}$.

Thus all focal menus can be found as maximal elements in the sets $X(\cdot, \succeq)$ or $X(\cdot, \preceq)$ when the primitives are the weak order $\preceq$ or dominance $\succeq$ respectively. Note that the focus $F(\preceq)$ is unaffected by comparisons between menus that differ in more than one element.\(^5\) Once the focus $F$ is found, the Boolean state space $B(F)$ can be written immediately.

Next, partition the focus $F$ into the minimal number of chains $D_1, \ldots, D_k$ where $k$ equals the width of the focus. Then a minimal state space can be identified as $S = \{R_1, \ldots, R_k\}$ where $R_i$ is represented by the utility function $\omega_{D_i}$.

\(^5\)Such comparisons are relevant for the axioms of the weak order $\succeq$ and dominance $\succeq$. 

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One can proceed further and attempt to turn the minimal state space \( S \) into a total one. To do so, each of the chains \( D_i \) should be extended to a total chain \( \tilde{D}_i \subset \pi(\mathcal{F}) \) so that \( D_i \subset \tilde{D}_i \). If such an extension exists, then it can be produced inductively. Suppose that \( \mathcal{D}'_i \subset \pi(\mathcal{F}) \) is a chain that extends \( D_i \), but is not total yet. Wlog, assume that \( Z \in \mathcal{D}'_i \). As \( \mathcal{D}'_i \) is not total, then there is \( A \in \mathcal{D}'_i \) such that \( |A| > 1 \), and there is no \( B \in \mathcal{D}'_i \) such that \( |B| = |A| - 1 \). Consider two cases.

- There is \( F \in \mathcal{F} \) such that \( |A \cap F| = |A| - 1 \) and \( \mathcal{D}'_i \cup \{A \cap F\} \) is a chain. Note that \( A \cap F \in \pi(\mathcal{F}) \) and replace \( \mathcal{D}'_i \) with the longer chain \( \mathcal{D}'_i \cup \{A \cap F\} \subset \pi(\mathcal{F}) \).

- There is no such \( F \in \mathcal{F} \). Then \( \succeq_1 \) cannot be represented by a total state space.

This induction will either extend \( D_i \) to a total \( \tilde{D}_i \subset \pi(\mathcal{F}) \) or terminate with the negative verdict.

Consider several examples. Let \( Z = \{a, b, c, d\} \).

**Example 3.1.** Let \( \succeq_1 \) be the induced dominance of \( \succeq_1 \) such that

\[
\begin{align*}
a &\sim_1 ab \succ_1 b \succ_1 Z \sim_1 abc \sim_1 acd \sim_1 bcd \sim_1 ac \sim_1 bc \sim_1 cd \sim_1 c \succ_1 \\
abd &\sim_1 ad \sim_1 bd \sim_1 d.
\end{align*}
\]

The maximal elements in the families \( \mathcal{X}(\cdot, \succeq_1) \) are marked in bold:

\[
\begin{align*}
\mathcal{X}(a, \succeq_1) &= \{b\} & \mathcal{X}(c, \succeq_1) &= \{abcd, ab, bd, ad, a, b, d\} \\
\mathcal{X}(b, \succeq_1) &= \emptyset & \mathcal{X}(d, \succeq_1) &= \{ab, a, b\}.
\end{align*}
\]

Thus \( \mathcal{F}(\succeq_1) = \mathcal{F}(\succeq_1) = \{abcd, ab, b\} \), and for all \( A, B \in \mathcal{M} \),

\[
A \succeq_1 B \iff AR_{ab}B \quad \text{and} \quad AR_{ab}B \quad \text{and} \quad AR_{b}B.
\]

The focus \( \mathcal{F}(\succeq_1) \) is a chain. Accordingly, \( \succeq_1 \) has a minimal state space \( S_1 = \{R\} \) where \( R \) is such that \( cPdPabPb \). For all \( A, B \in \mathcal{M} \),

\[
A \succeq_1 B \iff ARB.
\]

Thus \( \succeq_1 \) is the natural extension of \( R \) and hence, is complete. As \( R \) is total, then \( \succeq_1 \) is weakly total.

**Example 3.2.** Let \( \succeq_2 \) be induced by the preference

\[
\begin{align*}
Z &\sim_2 abc \sim_2 acd \sim_2 ac \succ_2 abd \succ_2 ad \succ_2 a \sim_2 ab \succ_2 \\
bc \sim_2 bd \succ_2 b \succ_2 cd \succ_2 d \succ_2 c.
\end{align*}
\]

The maximal elements in the families

\[
\begin{align*}
\mathcal{X}(a, \succeq_2) &= \{bcd, bc, bd, cd, b, c, d\} & \mathcal{X}(c, \succeq_2) &= \{abcd, ab, ad, bd, a, b, d\} \\
\mathcal{X}(b, \succeq_2) &= \{cd, c, d\} & \mathcal{X}(d, \succeq_2) &= \{ab, a, b, c\}
\end{align*}
\]

\[16\]
form the focus $F(\succeq_2) = F(\succeq_2) = \{bcd, abd, cd, ab, c\}$. Thus $\succeq_2$ is represented by the Boolean state space $B(F(\succeq_2)) = \{R_{bcd}, R_{abd}, R_{cd}, R_{ab}, R_c\}$.

The focus $F(\succeq_3)$ is not a chain, but it can be partitioned into two chains $D_1 = \{bcd, cd, c\}$ and $D_2 = \{abd, ab\}$. The corresponding minimal state space is $S = \{R_1, R_2\}$ where $aP_1bP_1dP_1c$ and $cP_2dP_2aP_2b$. The state space $S^*$ is total.

For all $A, B \in \mathcal{M}$,

$$A \succeq_2 B \iff AR_1B \text{ and } AR_2B \iff AR_1^*B \text{ and } AR_2^*B,$$

and $\succeq_2$ is weakly total.

**Example 3.3.** Let $\succeq_3$ be induced by the preference

$$Z \sim_3 abd \sim_3 acd \sim_3 ad \succ_3 bcd \sim_3 bd \sim_3 cd \succ_3 d \succ_3 abc \sim_3 ac \succ_3$$

$$bc \sim_3 c \succ_3 ab \succ_3 b \succ_3 a.$$

The maximal elements in the families

$$\mathcal{X}(a, \succeq_3) = \{bcd, bc, bd, cd, b, c, d\} \quad \mathcal{X}(c, \succeq_3) = \{ab, a, b, d\} \quad \mathcal{X}(b, \succeq_3) = \{a, d\} \quad \mathcal{X}(d, \succeq_3) = \{abc, ab, bc, ac, a, b, c\}$$

form the focus $F(\succeq_3) = \{bcd, abc, ab, a, d\}$. The focus $F(\succeq_3)$ is not a chain, but it can be partitioned into two disjoint chains $D_1 = \{bcd, d\}$ and $D_2 = \{abc, ab, a\}$. The corresponding minimal state space is $S = \{R_1, R_2\}$ where $aP_1bI_1cP_1d$ and $dP_2cP_2bP_2a$. Note that $D_1$ cannot be extended to a total chain $D_1^* \subset \pi(F(\succeq_3))$ because $bcd \cap abc = bc$ and $bcd \cap ab = b$ do not fill the gap between $bcd$ and $d$. For all $A, B \in \mathcal{M}$,

$$A \succeq_3 B \iff AR_1B \text{ and } AR_2B,$$

and $\succeq_3$ is not weakly total.

### 4 Subjective States via Preferences

Any complete and transitive preference $\succeq$ over menus can be modeled via a suitable aggregation of subjective states derived from its induced dominance $\succeq$. Say that $(U, \succeq, \succeq, F)$ is a preference tuple if $U : \mathcal{M} \to \mathbb{R}$ is a utility function, $\succeq$ is represented by $U$, $\succeq$ is induced by $\succeq$, and $F$ is the focus of $\succeq$. All components of the preference tuple are uniquely determined by $U$. If $\succeq$ is given, then $U$ can be any utility representation for $\succeq$.  

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4.1 Changing Tastes

The case of a singleton subjective state space delivers the two-period model of changing tastes that Gul and Pesendorfer [18] (henceforth GP) propose to accommodate time-inconsistent behaviors. In this model, the DM has normative utility $u \in \mathbb{R}^Z$ ex ante but expects to maximize a temptation utility $v \in \mathbb{R}^Z$ ex post. Her preferences are represented by

$$U(A) = u \left( \arg \max_{x \in A} v(x) \right) \quad \text{for all } A \in \mathcal{M}, \quad (22)$$

where $\arg \max_{x \in A} v(x)$ is the set of all maximizers of $v$ in $A$. If $v$ has more than one maximizer in $A$, then $u$ is used to break ex post ties.

GP establish the equivalence of representation (22) and

**Axiom 4** (No Compromise). For all $A, B \in \mathcal{M}$, either $A \sim A \cup B$ or $B \sim A \cup B$.

I refine GP’s characterization in several ways.

**Theorem 10.** For any preference tuple $(U, \succeq, \succeq, \mathcal{F})$, the following statements are equivalent:

(i) $\succeq$ satisfies No Compromise,

(ii) $\succeq$ is complete,

(iii) the focus $\mathcal{F}$ is a chain,

(iv) there are $u, v \in \mathbb{R}^Z$ such that (22) holds,

(v) there are $\omega \in \mathbb{R}^Z$ and $\phi : \omega \circ Z \to \mathbb{R}$ such that

$$U(A) = \phi(\omega(A)) \quad \text{for all } A \in \mathcal{M}, \quad (23)$$

(vi) there are $u, t \in \mathbb{R}^Z$ such that for all $A \in \mathcal{M}$,

$$U(A) = \max_{x \in A} \left[ u(x) - \max_{y \in A} (t(y) - t(x)) \right], \quad (24)$$

and $\mathcal{L}(\{u + t, t\})$ is a chain.

Moreover, representations (22)–(24) hold for

- $\omega = \omega_{\mathcal{F}}$, $v = \omega_{\mathcal{F}}$, and $u = U$ on $Z$,
- $\phi : \omega_{\mathcal{F}} \circ Z \to \mathbb{R}$ such that $\phi(\omega_{\mathcal{F}}(z)) = u(z)$ for all $z \in Z$,
- $t = \lambda \omega_{\mathcal{F}}$ where $\lambda = \max_{i=1,\ldots,|\mathcal{F}|} |\phi(i) - \phi(i-1)|$. 

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This result asserts that GP’s representation (22) is equivalent to either one of several conditions: No Compromise for \(\succeq\), completeness for \(\succeq\), the chain property for the focus \(\mathcal{F}\). Moreover, Theorem 10 provides two functional forms (23) and (24) that are also equivalent to GP’s model.

The decision maker as portrayed by (23) aggregates her singleton subjective state \(\omega\) via a function \(\phi\) that need not be monotonic. In contrast with the function \(u\) in (22), the function \(\phi\) is applied to the maximal value of \(\omega\) in the menu \(A\) rather than to the maximizers of \(\omega\) in \(A\). While the two models are equivalent in the finite framework, they become distinct when menus are defined as compact sets of lotteries. Then representation (23) is continuous for continuous \(\phi\) and \(v\), but (22) lacks continuity even if both \(u\) and \(v\) are regular expected utility functions.

Another observation is that in the finite framework, the model of changing tastes is a special case of Gul and Pesendorfer’s [17] model of costly self-control where the component

\[
\max_{y \in A}(t(y) - t(x))
\]

is interpreted as a cognitive cost of resisting the temptation \(y\) in favor of the consumption \(x\). This model can be rewritten as a two-state additive representation where \(w = u + t\) and

\[
U(A) = w(A) - t(A) \quad \text{for all } A \in \mathcal{M}.
\]

Note that the normative and temptation utilities \(u\) and \(t\) in (24) are not arbitrary: the lower contours sets \(L(w) = L(u+t)\) and \(L(t)\) must form a single chain together.

Moreover, Theorem 10 derives all representations (22)–(24) explicitly in terms of the focus \(\mathcal{F}\) and the normative utility index \(u : Z \to \mathbb{R}\), which equals \(U\) on \(Z\).

Recall Example 3.1 where the preference \(\succeq_1\) is represented by

\[
\begin{align*}
U_1(a) &= U_1(ab) = 3 \\
U_1(b) &= 2 \\
U_1(Z) &= U_1(abc) = U_1(acd) = U_1(bcd) = U_1(ac) = U_1(bc) = U_1(cd) = U_1(c) = 1 \\
U_1(abd) &= U_1(ad) = U_1(bd) = U_1(d) = 0.
\end{align*}
\]

The focus \(\mathcal{F} = \{abd, ab, b\}\) is a chain and the minimal subjective state space is the singleton \(S = \{R\}\) where \(R\) is represented by \(\omega_\mathcal{F}\) such that \(\omega_\mathcal{F}(c) = 3 > \omega_\mathcal{F}(d) = 2 > \omega_\mathcal{F}(a) = 1 > \omega_\mathcal{F}(b) = 0\). Let \(\omega = v = \omega_\mathcal{F}\) and \(u = U_1\) on \(Z\) so that \(u(a) = 3\), \(u(b) = 2\), \(u(c) = 1\), \(u(d) = 0\). The corresponding \(\phi : \{0,1,2,3\} \to \mathbb{R}\) is

\[
\begin{align*}
\phi(0) &= u(b) = 2 & \phi(1) &= u(a) = 3 \\
\phi(2) &= u(d) = 0 & \phi(3) &= u(c) = 1.
\end{align*}
\]

The corresponding

\[
\lambda = \max\{|\phi(1) - \phi(0)|, |\phi(2) - \phi(1)|, |\phi(3) - \phi(2)|\} = 3.
\]
Take $t = 3\omega$. Then

\[
\begin{array}{ccc}
  & a & b & c & d \\
 t & 3 & 0 & 9 & 6 \\
u + t & 6 & 2 & 10 & 6
\end{array}
\]

Note that for all menus $A \in \mathcal{M}$,

\[
U_1(A) = u\left(\text{Arg max}_{x \in A} v(x)\right) = \phi(\omega(A)) = \max_{x \in A} \left[u(x) - \max_{y \in A}(t(y) - t(x))\right].
\]

4.2 Coherent Aggregation

Take any preference tuple $(U, \succeq, \preceq, \mathcal{F})$. In general, the state spaces $S \subset \mathbb{R}$ derived from the induced dominance $\succeq$ need not be singleton, but they can be still used to model the preference $\succeq$ and the utility function $U$. This model is called coherent aggregation and requires some additional terminology.

In aggregation problems, state spaces $S \subset \mathbb{R}$ are replaced by sets of functions $\Omega \subset \mathbb{R}^Z$. Say that $\succeq$ is represented by a set $\Omega \subset \mathbb{R}^Z$ if for all $A, B \in \mathcal{M}$,

\[
A \succeq B \iff \omega(A) \geq \omega(B) \quad \text{for all } \omega \in \Omega.
\]

Equivalently, $\succeq$ is represented by

\[
\mathcal{R}_\Omega = \{R \in \mathcal{R} : R \text{ is represented by some } \omega \in \Omega\}.
\]

Fix any $\Omega \subset \mathbb{R}^Z$. Let $\mathcal{L}(\Omega) = \mathcal{L}(\mathcal{R}_\Omega)$ be the collection of all lower contour sets of the functions $\omega \in \Omega$. For any $A \in \mathcal{M}$, let

- $\Omega(A)$ be the vector $s : \Omega \to \mathbb{R}$ such that $s(\omega) = \omega(A)$ for all $\omega \in \Omega$; say that $\Omega(A)$ is induced by $A$;
- $\Omega \circ \mathcal{M} = \{\Omega(A) : A \in \mathcal{M}\}$ be the range of all vectors $\Omega(A)$ that can be induced by some menu $A \in \mathcal{M}$.

For any vectors $s, t \in \Omega \circ \mathcal{M}$, write

- $s \succeq t$ if $s(\omega) \geq t(\omega)$ for all $\omega \in \Omega$,
- $s \succ t$ if $s(\omega) \geq t(\omega)$ for all $\omega \in \Omega$, and $s(\omega) > t(\omega)$ for some $\omega \in \Omega$,
- $s \vee t$ if $(s \vee t)(\omega) = \max\{s(\omega), t(\omega)\}$ for all $\omega \in \Omega$.

For all $A, B \in \mathcal{M}$, $\Omega(A) \vee \Omega(B) = \Omega(A \cup B)$. Thus $\Omega \circ \mathcal{M}$ is a semilattice:

\[
s, t \in \Omega \circ \mathcal{M} \Rightarrow s \vee t \in \Omega \circ \mathcal{M}.
\]

Functions $\phi : \Omega \circ \mathcal{M} \to \mathbb{R}$ are called aggregators. For any set $\Omega \subset \mathbb{R}^Z$ and aggregator $\phi : \Omega \circ \mathcal{M} \to \mathbb{R}$, define the composition $\phi \circ \Omega : \mathcal{M} \to \mathbb{R}$

\[
(\phi \circ \Omega)(A) = \phi(\Omega(A)) \quad \text{for all } A \in \mathcal{A}.
\]

An aggregator $\phi : \Omega \circ \mathcal{M} \to \mathbb{R}$ is called
- **monotonic** if for all \( s, t \in \Omega \circ M \), \( s \geq t \) implies \( \phi(s) \geq \phi(t) \),

- **submodular** if for all \( s, t \in \Omega \circ M \), \( s > t \) implies \( \phi(s) \neq \phi(t) \),

- **strictly monotonic** if for all \( s, t \in \Omega \circ M \), \( s > t \) implies \( \phi(s) > \phi(t) \); this condition is equivalent to the combination of monotonicity and submodularity;

- **coherent** if for all \( s, s' \in \Omega \circ M \),

\[
s \neq s' \implies \phi(s \vee t) \neq \phi(s' \vee t) \quad \text{for some } t \in \Omega \circ M.
\]

If \( \phi \) is submodular, then for all \( s, s' \in \Omega \circ M \),

\[
s \neq s' \implies s \vee s' > s' \vee s' \implies \phi(s \vee t) > \phi(s' \vee t) \quad \text{for } t = s'.
\]

Thus submodularity implies coherence.

**Theorem 11.** For any preference tuple \((U, \succeq, \preceq, \mathcal{F})\) and space \(\Omega \subset \mathbb{R}^Z\), the following statements are equivalent:

(i) \( \mathcal{F} \subset \mathcal{L}(\Omega) \subset \pi(\mathcal{F}) \),

(ii) \( \preceq \) is represented by \(\Omega\),

(iii) \( U = \phi \circ \Omega \) for some coherent aggregator \( \phi : \Omega \circ M \to \mathbb{R} \).

The function \( \phi \) that satisfies \( U = \phi \circ \Omega \) is

- **unique**, 

- **monotonic if and only if** \( \succeq \) **is monotonic**, 

- **submodular if and only if** \( \preceq \) **is submodular**.

The equivalence of statements (i) and (ii) in Theorem 11 is the same as in Theorem 3. The novel claim is that any utility function \( U \) that represents the primitive preference \( \preceq \) can be obtained via a unique coherent aggregator \( \phi \) from any set \( \Omega \subset \mathbb{R}^Z \) that represents the induced dominance \( \preceq \). Moreover, the aggregator \( \phi \) must inherit monotonicity and submodularity from the preference \( \succeq \) when it satisfies one or both of these properties. In particular, \( \phi \) is strictly monotonic if and only if \( \succeq \) is monotonic and submodular.

Consider some examples. Recall the preference \( \succeq_2 \) from Example 3.2. It is represented by the function \( U_2 : M \to \mathbb{R} \) such that

\[
U_2(Z) = U_2(abc) = U_2(acd) = U_2(ac) = 8 > U_2(abd) = U_2(ad) = 7 > \\
U_2(a) = U_2(ab) = 6 > U_2(bcd) = U_2(bc) = 5 > \\
U_2(bd) = 4 > U_2(b) = 3 > U_2(cd) = 2 > U_2(d) = 1 > U_2(c) = 0.
\]
The induced dominance $\succeq_2$ is represented by $\Omega = \{\omega_1, \omega_2\}$ such that

\[
\begin{array}{cccc}
a & b & c & d \\
\omega_1 & 3 & 2 & 0 & 1 \\
\omega_2 & 1 & 0 & 3 & 2 \\
\end{array}
\]

By Theorem 11, $U_2 = \phi \circ \Omega$ with the unique coherent aggregator $\phi$. It satisfies

\[
\begin{array}{c}
\phi(0, 3) = 0 \\
\phi(1, 2) = 1 \\
\phi(1, 3) = 2 \\
\phi(2, 0) = 3 \\
\phi(2, 2) = 4 \\
\phi(2, 3) = 5 \\
\phi(3, 1) = 6 \\
\phi(3, 2) = 7 \\
\phi(3, 3) = 8.
\end{array}
\]

The function $\phi$ is strictly monotonic.

Consider a more insightful situation.

**Example 4.1.** Let $\succeq_4$ be induced by the Krepsian utility function

\[
U_4(A) = \omega'_1(A) + \omega'_2(A) + \omega'_3(A)
\]

where the functions $\omega'_1, \omega'_2, \omega'_3 \in \mathbb{R}^Z$ are such that

\[
\begin{array}{cccc}
a & b & c & d \\
\omega'_1 & 13 & 10 & 7 & 0 \\
\omega'_2 & 0 & 5 & 10 & 12 \\
\omega'_3 & 3 & 3 & 0 & 3 \\
\end{array}
\]

By Proposition 2, $\succeq_4$ is represented by $S = \{R'_1, R'_2, R'_3\}$ where $R'_i$ is represented by $\omega'_i$. By Theorem 9, the maximal menus in the families

\[
\begin{align*}
\mathcal{X}(a, S) &= \{bcd, \emptyset, c\} \\
\mathcal{X}(b, S) &= \{cd, a, c\} \\
\mathcal{X}(c, S) &= \{d, ab, \emptyset\} \\
\mathcal{X}(d, S) &= \{\emptyset, abc, c\}
\end{align*}
\]

form the focus $\mathcal{F}(\succeq_4) = \mathcal{F}(S) = \{bcd, cd, d, a, ab, abc\}$.

The focus $\mathcal{F}(\succeq_4)$ can be partitioned into two disjoint chains $\mathcal{D}_1 = \{bcd, cd, d\}$ and $\mathcal{D}_2 = \{abc, ab, a\}$. By Theorem 11, $\succeq_4$ is represented by $\Omega = \{\omega_1, \omega_2\}$ such that

\[
\begin{array}{cccc}
a & b & c & d \\
\omega_1 & 3 & 2 & 1 & 0 \\
\omega_2 & 0 & 1 & 2 & 3 \\
\end{array}
\]

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and $U_4 = \phi \circ \Omega$ for a unique $\phi : \Omega \circ M \to \mathbb{R}$. The aggregator $\phi$ is coherent and satisfies

\[
\begin{align*}
\phi(0, 3) &= 15 \\
\phi(1, 2) &= 17 \\
\phi(1, 3) &= 22 \\
\phi(2, 1) &= 18 \\
\phi(2, 2) &= 23 \\
\phi(2, 3) &= 25 \\
\phi(3, 0) &= 16 \\
\phi(3, 1) &= 21 \\
\phi(3, 2) &= 26 \\
\phi(3, 3) &= 28.
\end{align*}
\]

The function $\phi$ is strictly monotonic.

By contrast, $\succeq_4$ does not have any two-state additive representation. Indeed, suppose that there are $\omega^*_1, \omega^*_2 \in \mathbb{R}^Z$ such that $\succeq_4$ is represented by

\[U^*(A) = \omega^*_1(A) + \omega^*_2(A)\]

for all $A \in M$. The additive aggregation is coherent. By Theorem 11,

\[\mathcal{F}(\succeq_4) \subset \mathcal{L}(\omega^*_1) \cup \mathcal{L}(\omega^*_2)\]

There is only one way to cover $\mathcal{F}(\succeq_4)$ by two chains of lower contour sets,

\[\mathcal{F}(\succeq_4) \subset \{d, cd, bcd, Z\} \cup \{a, ab, abc, Z\}\]

Wlog $\omega^*_1(a) > \omega^*_1(b) > \omega^*_1(c) > \omega^*_1(d)$ and $\omega^*_2(d) > \omega^*_2(c) > \omega^*_2(b) > \omega^*_2(a)$, and

\[
\begin{align*}
\omega^*_1(a) + \omega^*_2(c) &> \omega^*_1(b) + \omega^*_2(d) \quad \text{because } ac \succ_4 bd \\
\omega^*_1(c) + \omega^*_2(d) &> \omega^*_1(a) + \omega^*_2(b) \quad \text{because } cd \succ_4 ab \\
\omega^*_1(b) + \omega^*_2(b) &> \omega^*_1(c) + \omega^*_2(c) \quad \text{because } b \succ_4 c.
\end{align*}
\]

Adding these inequalities leads to a contradiction.

### 4.3 Combinatorial Extensions

The coherent aggregation of minimal state spaces can be applied to monotonic preferences $\succeq_k$ over menus of limited size.

Fix any $k \in \{1, \ldots, |Z|\}$. Let $M^k$ be the family of all menus $A \in M$ such that $|A| \leq k$. A function $V : M^k \to \mathbb{R}$ is called monotonic if $V(B) \geq V(A)$ for all $A, B \in M^k$ such that $B \supseteq A$. Obviously, a monotonic utility function can be obtained for any monotonic, complete and transitive preference $\succeq_k$ on $M^k$.

Say that $(V, U, \succeq, \approx, \mathcal{F})$ is a combinatorial tuple if

- $V : M^k \to \mathbb{R}$ is monotonic,
- $U : M \to \mathbb{R}$ is such that for all $A \in M$,

\[
U(A) = \max_{D \in M^k : D \subseteq A} V(D)
\]

(27)
• $(U, \succeq, \geq, \mathcal{F})$ is a preference tuple.

By construction, $U$ is monotonic and $U = V$ on $\mathcal{M}^k$.

**Corollary 12.** For any combinatorial tuple $(V, U, \succeq, \geq, \mathcal{F})$, the following statements are equivalent:

(i) the width of $\mathcal{F}$ does not exceed $k$,

(ii) $V = \phi \circ \Omega$ where the set $\Omega \subset \mathbb{R}^Z$ is such that $|\Omega| \leq k$, and $\phi : \Omega \circ \mathcal{M} \rightarrow \mathbb{R}$ is a monotonic, coherent aggregator.

**Proof.** The proof is constructive. Take any combinatorial tuple $(V, U, \succeq, \geq, \mathcal{F})$. It can be derived from $V$ via (27). Suppose that the width of $\mathcal{F}$ does not exceed $k$. Partition $\mathcal{F}$ into $k$ chains $D_1, \ldots, D_k$. (Empty chains are allowed). Let $\Omega$ be the set of all functions $\omega_{D_i}$ for $i = 1, \ldots, k$. By Theorem 11, $U = \phi \circ \Omega$ for some $\phi : \Omega \circ \mathcal{M} \rightarrow \mathbb{R}$ that is monotonic and coherent. As $\Omega \circ \mathcal{M} = \Omega \circ \mathcal{M}^k$, then $V = \phi \circ \Omega$.

Conversely, if $V = \phi \circ \Omega$ and $\Omega$ has at most $k$ states, then $\Omega \circ \mathcal{M} = \Omega \circ \mathcal{M}^k$ and hence, $U = \phi \circ \Omega$. By Theorem 11, $\Omega$ represents $\succeq$ and hence, the width of $\mathcal{F}$ does not exceed $k$. \qed

To illustrate, consider the examples from the introduction. Take $Z = \{a, b, c\}$ and take $V$ on $\mathcal{M}^2$ such that

$$V(bc) = 4 > V(ac) = 3 > V(ab) = V(a) = 2 > V(b) = 1 > V(c) = 0.$$  

The corresponding $\succeq$ is

$$Z \sim bc \succ ac \succ ab \sim a \succ c \succ b.$$  

The maximal menus in the families

$$\mathcal{X}(a, \succeq) = \{b, c\} \quad \mathcal{X}(b, \succeq) = \{ac\} \quad \mathcal{X}(c, \succeq) = \{ab\}$$

form the focus $\mathcal{F} = \{ab, ac, b, c\}$. The width of $\mathcal{F}$ is two. Partition $\mathcal{F}$ into two chains $D_1 = \{ac, c\}$ and $D_2 = \{ab, b\}$. Let $\Omega = \{\omega_1, \omega_2\}$ where $\omega_i = \omega_{D_i}$ are

$$\begin{array}{ccc}
   a & b & c \\
   \omega_1 & 1 & 2 & 0 \\
   \omega_2 & 1 & 0 & 2.
\end{array}$$

Then $V = \phi \circ \Omega$ where

$$\phi(2, 2) = 4 > \phi(1, 2) = 3 > \phi(2, 1) = \phi(1, 1) = 2 > \phi(0, 2) = 1 > \phi(2, 0) = 0.$$  

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Note that the subjective state space $S = R_\Omega$ consists of the rankings $R_1$ and $R_2$ such that $bP_1aP_1c$ and $cP_2aP_2b$.

Recall Example 4.1 where $Z = \{a, b, c, d\}$. Take $V(A) = \omega'_1(A) + \omega'_2(A) + \omega'_3(A)$ for all $A \in M^2$. The corresponding preference $\succeq$ is

$$Z \sim abd \sim acd \sim ad \succ abc \sim ac \succ bcd \sim bd \succ cd \succ ab \succ b \succ c \succ a \succ d.$$  

This preference equals $\succeq_4$ in Example 4.1. Thus $V = \phi \circ \Omega$ for the same $\phi$ and $\Omega$ obtained for $U_4$ in Example 4.1. Note that there is no two-state additive representation for $U_4$. Thus there is no two-state additive representation for $V$ either, even though $\succeq$ does satisfy the Krepsian axioms.

Combinatorial applications like Corollary 12 can be generalized for preferences and functions $V$ given over menus with precisely $k$ elements. In this case, the function $U$ can be defined via (27) over the domain $M_k$ of all menus that have at least $k$ elements. The preference $\succeq$, the induced dominance $\succeq$, and the focus $F$ can be redefined accordingly.

For example, let $Z = \{a, b, c, d\}$ and $V : M^2 \to \mathbb{R}$ such that $V(ab) = 5 > V(cd) = 4 > V(ac) = 3 > V(bd) = 2 > V(ad) = 1 > V(bc) = 0$.

The corresponding preference $\succeq$ is

$$Z \sim abc \sim abd \sim ab \succ acd \sim bcd \sim cd \succ ac \succ bd \succ ad \succ bc.$$  

The focus $F = \{acd, bcd, ad, bd, ac, bc\}$. Its width is four because $\{ad, bc, ac, bc\}$ is an antichain. Thus one cannot model $V$ by coherent aggregation of two states.

### 4.4 Other Aggregation Models

In the joint paper with Zhao, I study two-state additive aggregation models that include representations

$$U(A) = \max_{z \in A} w(z) + \max_{z \in A} v(z)$$  

$$U(A) = \max_{z \in A} w(z) - \max_{z \in A} v(z)$$  

where $w, v \in \mathbb{R}^Z$. Representation (29) is the additive model of costly self-control that Gul and Pesendorfer [17] characterize for preferences over menus of lotteries. For finite menus, the two states in both models can be identified via focal menus, but the additivity requires substantial additional effort.

A special case of (28) is the lexicographic representation

$$A \succeq B \iff w(A) > w(B) \text{ or } w(A) = w(B) \text{ and } v(A) \geq v(B)$$  

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for some \( w, v \in \mathbb{R}^Z \). Analogously to Manzini and Mariotti [26], representation (30) can be interpreted in terms of sequential rationalizability, where the decision maker first selects all menus that maximize the function \( w \) as acceptable, and then maximizes \( v \) to choose among acceptable menus. The lexicographic model is a special case of the additive model (28), and is much easier to characterize. Both (28) and (30) can be applied to combinatorial extensions of preferences over menus that have at most two elements.

5 Subjective States via Choice Functions

Dominance relations and subjective state spaces have natural counterparts for choice functions in the standard framework of Uzawa [31] and Arrow [3].

Say that \( C : \mathcal{M} \to 2^Z \) is a choice function if \( C(A) \subset A \) for all \( A \in \mathcal{M} \) and \( C(z) = z \) for all \( z \in Z \). Note that \( C(A) \) is allowed to be empty for non-singleton menus \( A \).

For any dominance \( \succeq \), its essential choice function \( C \) is defined as

\[
C(A) = \{ z \in A : A \succ A \setminus z \} \tag{31}
\]

for all non-singleton menus \( A \in \mathcal{M} \). By convention, let \( C(z) = z \) for all \( z \in Z \). Note that \( C(A) \subset A \) for all \( A \in \mathcal{M} \), and \( C(A) \) can be empty for non-singleton \( A \). Puppe [30] uses condition (31) to define essential elements in the Krepsian model.

Essential choice functions must obey the following pair of axioms.

**Axiom 5 (Substitution).** For all \( A \in \mathcal{M} \) and \( x, y \in Z \),

\[
x \in C(A) \text{ and } y \notin C((A \setminus x) \cup y) \Rightarrow x \in C(A \cup y).
\]

**Axiom 6 (Chernoff).** For all \( A, B \in \mathcal{M} \) and \( z \in A \),

\[
z \in C(A \cup B) \Rightarrow z \in C(A).
\]

Suppose that \( x \in C(A) \) and \( y \notin C((A \setminus x) \cup y) \). Then \( x \neq y \) and \( x \in C(A \cup y) \):

\[
A \cup y \succeq A \Rightarrow A \setminus x \congruent (A \setminus x) \cup y = (A \cup y) \setminus x.
\]

Turn to Chernoff. Suppose that \( z \notin C(A) \). Then \( A \setminus z \congruent A \). By additivity, \((A \setminus z) \cup (B \setminus z) \congruent A \cup (B \setminus z) = A \cup B \). Thus \( z \notin C(A \cup B) \). The Chernoff condition is also known as Sen’s property \( \alpha \).

**Theorem 13.** A choice function \( C \) satisfies Chernoff and Substitution if and only if \( C \) is essential for some dominance \( \succeq \).

Moreover, such \( \succeq \) is unique. For all menus \( A, B \in \mathcal{M} \), it satisfies

\[
A \succeq B \iff z \notin C(A \cup z) \text{ for all } z \in B \setminus A. \tag{32}
\]
This result characterizes all essential choice functions and reconstructs any dominance $\succeq$ uniquely in terms of its essential choice function.\footnote{The latter point is due to Danilov et al. [7].}

Consider another well-known condition, due to Plott [29]:

**Axiom 7** (Path Independence). For all $A, B \in \mathcal{M}$,

$$C(A \cup B) = C(C(A) \cup B).$$

This condition implies that for all $A, B \in \mathcal{M}$,

- $C(A)$ is not empty: if $C(A) = \emptyset$ then for any $y \in A$,
  $$C(A) = C(A \cup y) = C(C(A) \cup y) = C(y) = y,$$

- $C$ satisfies Chernoff: if $z \in C(A \cup B)$ and $z \in A$, then $z \in C(C(A) \cup (B \setminus A))$ and hence, $z \in C(A)$,

- $C$ satisfies Substitution: if $x \in C(A)$ and $y \notin C((A \setminus x) \cup y)$, then
  $$C(A \cup y) = C(A \cup C((A \setminus x) \cup y))$$
  and $x \in C(A \cup y)$ because $C((A \setminus x) \cup y) \subset A$.

Path Independence characterizes a subclass of essential choice functions that corresponds to weakly total dominance relations.

**Theorem 14.** A choice function $C$ is path independent if and only if $C$ is essential for some weakly total dominance $\succeq$.

Theorems 13 and 14 allow to translate choice functions $C$ into dominance relations $\succeq$ and hence, represent $C$ via subjective states that are derived from $\succeq$.

For any $R \in \mathcal{R}$ and $A \in \mathcal{M}$, let

- $\max_A R$ be the set of all *strictly maximal* elements $x \in A$ such that $xPy$ for all $y \in A \setminus x$,

- $\text{Max}_A R$ be the set of all *maximal* elements $x \in A$ such that $xRy$ for all $y \in A$.

Obviously, $\max_A R$ is either singleton or empty, $\text{Max}_A R$ is not empty, and $\max_A R \subset \text{Max}_A R$. If $R$ is total, then $\max_A R = \text{Max}_A R$.

A choice function $C$ is rationalized by a state space $S \subset \mathcal{R}$ if for all $A \in \mathcal{M}$,

$$C(A) = \bigcup_{R \in S} \max_A R.$$  \hspace{1cm} (33)
In particular, $C$ is rationalized by a total state space $S \subset \mathcal{R}$ if
\[
C(A) = \bigcup_{R \in S} \text{Max}_A R.
\] (34)

Aizerman and Malishevski [2] show that $C$ is rationalized by a total state space $S \subset \mathcal{T}$ if and only if $C$ is path independent. This finding can be extended to arbitrary state spaces.

**Proposition 15.** A choice function $C$ is rationalized by $S \subset \mathcal{R}$ if and only if $C$ is essential for the dominance $\geq$ represented by $S$.

**Proof.** Suppose that $C$ is rationalized by $S \subset \mathcal{R}$. Let $C^*$ be the essential choice function for $\geq$ that is represented by $S$. Take any menu $A \in \mathcal{M}$. Take any $z \in C(A)$. Then there is $R \in S$ such that $zPx$ for all $x \in A \setminus z$. As $S$ represents $\geq$, then $A \setminus z \geq z$ does not hold. Thus $A \gg A \setminus z$ and $z \in C^*(A)$. Conversely, take any $z \in C^*(A)$ and retrace the previous claims in the opposite direction to show that $z \in C(A)$. Thus $C = C^*$. \qed

The above results imply that

1. $C$ satisfies Chernoff and Substitution if and only if $C$ is rationalized by some $S \subset \mathcal{R}$,

2. $C$ is path independent if and only if $C$ is rationalized by some total state space $S \subset \mathcal{T}$,

3. the minimal number of states in the subjective state space $S$ in representations (33) and (34) equals the width of the focus $\mathcal{F}$ of the dominance relation $\geq$ for which $C$ is essential.

By Theorem 9, the focus $\mathcal{F}$ of the dominance $\geq$ for which $C$ is essential consists of all maximal elements in the families
\[
\mathcal{X}(y, C) = \mathcal{X}(y, \geq) = \{ A \in \mathcal{M} : y \in C(A \cup y) \}
\]
for all $y \in Z$.

To illustrate, take $Z = \{a, b, c, d\}$ and a choice function $C$ such that
\[
C(Z) = ac \quad C(abc) = ac \quad C(acd) = ac \quad C(abd) = ad \quad C(bcd) = bc \quad C(ab) = a \quad C(ac) = ac \\
C(ad) = ad \quad C(bc) = bc \quad C(bd) = bd \quad C(cd) = cd.
\]

The maximal elements in the families
\[
\mathcal{X}(a, C) = \{bcd, bc, bd, cd, b, c, d\} \quad \mathcal{X}(c, C) = \{abd, ab, ad, bd, a, b, d\} \\
\mathcal{X}(b, C) = \{cd, c, d\} \quad \mathcal{X}(d, C) = \{ab, a, b, c\}
\]
form the focus $\mathcal{F} = \{bcd, abd, ab, cd, ab, c\}$. The dominance $\geq_2$ in Example 3.2 has the same focus. Note that $C$ is essential for $\geq_2$ because $\mathcal{X}(z, C) = \mathcal{X}(z, \geq_2)$ for all $z \in Z$. Thus $C$ is rationalized by the total state space $S = \{R^*_1, R^*_2\}$ where $R^*_1 = aP^*_1bP^*_1cP^*_1d$ and $cP^*_2dP^*_2aP^*_2b$. Path independence is implied.
5.1 Combinatorial Extensions

Subjective state spaces of size up to \( k \) can be derived from choices in menus of size up to \( k + 1 \). To do so, take any \( C : \mathcal{M}^{k+1} \rightarrow 2^Z \) such that \( C(A) \subset A \) for all \( A \in \mathcal{M}^{k+1} \) and \( C(z) = z \) for all \( z \in Z \).

Let the combinatorial extension of \( C \) be the choice function \( \mathcal{E} \) such that

\[
\mathcal{E}(A) = \{ z \in A : z \in C(B) \text{ for all } B \in \mathcal{M}^{k+1} \text{ such that } z \in B \subset A. \}
\]

Check whether \( \mathcal{E} \) can be rationalized by \( S \) such that \( |S| \leq k \). If so, then \( C \) can be rationalized by \( S \) as well. Moreover, if \( \mathcal{E} \) is path independent, then \( S \subset \mathcal{T} \) can be taken total. If \( \mathcal{E} \) cannot be rationalized by \( k \) subjective states, then it cannot be done for \( C \) either.

For example, let \( Z = \{ a, b, c, d \} \) and the choice function \( C \) on \( \mathcal{M}^3 \) be such that

\[
\begin{align*}
C(abc) &= ac & C(acd) &= ad & C(abd) &= ad & C(bcd) &= d \\
C(ab) &= ab & C(ac) &= ac & C(ad) &= ad & C(bc) &= c & C(bd) &= bd & C(cd) &= cd.
\end{align*}
\]

The combinatorial extension \( \mathcal{E} \) from \( \mathcal{M}^3 \) to \( \mathcal{M} \) assigns \( \mathcal{E}(Z) = ad \) and \( \mathcal{E} = C \) on \( \mathcal{M}^3 \). The maximal elements in the families

\[
\begin{align*}
\mathcal{X}(a, \mathcal{E}) &= \{ bcd, bc, bd, cd, b, c, d \} & \mathcal{X}(c, \mathcal{E}) &= \{ ab, a, b, d \} \\
\mathcal{X}(b, \mathcal{E}) &= \{ a, d \} & \mathcal{X}(d, \mathcal{E}) &= \{ abc, ab, bc, ac, a, b, c \}
\end{align*}
\]

form the focus \( \mathcal{F} = \{ bcd, abc, ab, a, d \} \). Thus \( \mathcal{E} \) is essential for \( \geq 3 \) because \( \mathcal{X}(z, \mathcal{E}) = \mathcal{X}(z, \geq 3) \) for all \( z \in Z \). It follows that \( \mathcal{E} \) and \( C \) are rationalized by \( S = \{ R_1, R_2 \} \) such that \( aP_1bI_1cP_1d \) and \( dP_2cP_2bP_2a \). As \( \geq 3 \) is not weakly total, then \( E \) is not path independent.

5.2 Choice Samples and Dominance Hulls

In practice, it is rarely possible to collect choice data in all menus in \( \mathcal{M} \), especially when \( Z \) is not small. To start with more realistic primitives, suppose that the choice function \( C : \mathcal{D} \rightarrow 2^Z \) is observed on a subdomain \( \mathcal{D} \subset \mathcal{M} \). Wlog, assume that all singletons belong to \( \mathcal{D} \), and \( C(z) = z \) for all \( z \in Z \). As customary, assume that \( C(A) \subset A \) for all \( A \in \mathcal{D} \).

Call the pair \( (C, \mathcal{D}) \) a choice sample. Say that \( (C, \mathcal{D}) \) is rationalized by \( S \subset \mathcal{R} \) if for all \( A \in \mathcal{D} \),

\[
C(A) = \bigcup_{R \in S} \max_A R.
\]

The existence of such rationalizations can be established constructively via suitable dominance relations.
For any binary relation $\succ$ on $\mathcal{M}$, its dominance hull $\succeq = \text{dom} (\succ)$ is the minimal dominance relation that contains $\succ$:

$$\succeq = \bigcap \{ \succeq^* : \succeq^* \supseteq \succ \} \text{ and } \succeq^* \text{ is a dominance relation } \}.$$ 

This overlap is not empty because the trivial dominance relation $\succeq^* = \mathcal{M} \times \mathcal{M}$ contains any $\succ$. Moreover, $\succeq$ is a dominance relation because taking intersections preserves monotonicity, transitivity, and additivity.

For any choice sample $(\mathcal{C}, \mathcal{D})$, let $\mathcal{C}(A) \succ A$ for all $A \in \mathcal{D}$. Let $\succeq$ be the dominance hull of $\succ$, and let $\mathcal{E}$ be the essential choice function of $\succeq$.

**Proposition 16.** A choice sample $(\mathcal{C}, \mathcal{D})$ is rationalized by some state space $\mathcal{S} \subset \mathcal{R}$ if and only if for all $A \in \mathcal{D}$,

$$\mathcal{C}(A) = \mathcal{E}(A). \quad (35)$$

**Proof.** The dominance hull $\succeq$ is represented by some $\mathcal{S} \subset \mathcal{R}$. By Proposition 15, $\mathcal{E}$ is also rationalized by $\mathcal{S}$. If $(35)$ holds, then $\mathcal{C}$ is rationalized by the same $\mathcal{S}$.

Suppose instead that $\mathcal{C}$ is rationalized by some $\mathcal{S}$. Show $(35)$. Take any non-singleton $A \in \mathcal{M}$ and $z \in A$. Assume $z \notin \mathcal{C}(A)$. As $\mathcal{C}(A) \supseteq A$ and $\succeq$ extends $\succ$, then $\mathcal{C}(A) \supseteq A \succeq A \setminus z \succeq \mathcal{C}(A)$. Thus $A \setminus z \cong A$ and $z \notin \mathcal{E}(A)$. Assume $z \in \mathcal{C}(A)$. Then there is $R \in \mathcal{S}$ such that $z \in \max_A R$. Let $\succeq^*$ be the dominance represented by $\mathcal{S}$. Then $A \setminus z \succeq^* z$ does not hold. As $\succeq \subseteq \succeq^*$, then $A \setminus z \succeq z$ does not hold either. Thus $z \in \mathcal{E}(A)$.

Proposition (16) shows that the existence and the construction of subjective states for a choice sample $(\mathcal{C}, \mathcal{D})$ can be reduced to finding the dominance hull $\succeq$, which can be further analyzed in terms of the focus $\mathcal{F}$.

For example, let $\mathcal{Z} = \{a, b, c, d\}$, $\mathcal{D} = \{abc, bcd, abd, acd\}$ and

$$\mathcal{C}(abc) = ab \quad \mathcal{C}(bcd) = cd \quad \mathcal{C}(abd) = bd \quad \mathcal{C}(acd) = ac.$$ 

Then $ab \succ abc$, $cd \succ bcd$, $bd \succ abd$, and $ac \succ acd$. As $ab$ dominates $c$ and $ac$ dominates $d$, then $ab \cong Z$. As $cd$ dominates $b$ and $bd$ dominates $a$, then $cd \cong Z$. As $bd$ dominates $a$ and $ab$ dominates $c$, then $bd \cong Z$. As $ac$ dominates $d$ and $cd$ dominates $b$, then $ac \cong Z$. A fortiori, $A \cong Z$ for all three element $A$. The sets $ad$ and $bc$ dominate only their own subsets. Singletons dominate only themselves.

Thus $\mathcal{E}(abc) = a$ because $abc \succ bc$, but $abc \cong ac \cong ab$. As $\mathcal{E}(abc) \neq \mathcal{C}(abc)$, then $(\mathcal{C}, \mathcal{D})$ is not rationalized by any $\mathcal{S}$.

Suppose instead that

$$\mathcal{C}(abc) = ac \quad \mathcal{C}(bcd) = bd \quad \mathcal{C}(abd) = ad \quad \mathcal{C}(acd) = ad.$$ 

Then $ad \cong acd \cong abd \cong Z$, $Z \succ ac \cong abc$, and $Z \succ bd \cong bcd$. The focus $\mathcal{F}$ is $\{abc, bcd, cd, a, ab, d\}$. Thus $\mathcal{C}$ is rationalized by $\mathcal{S} = \{R_1, R_2\}$ such that $\mathcal{L}(R_1) = \{bcd, cd, d\}$ and $\mathcal{L}(R_2) = \{abc, ab, a\}$.

The computation of dominance hulls and their applications to choice functions are further discussed in Kopylov [23].
A APPENDIX: PROOFS

Let $\geq$ be a dominance relation on $\mathcal{M}$. Then $\geq$ is monotonic, transitive, and additive. These properties imply that for all $A, B, D \in \mathcal{M}$,

$$
A \geq B \iff A \geq A \cup B \iff A \equiv A \cup B \tag{36}
$$

$$
A \geq B \iff A \geq x \quad \text{for all } x \in B, \tag{37}
$$

$$
A \cup B \cup D \supseteq B \cup D \implies A \cup D \supseteq D. \tag{38}
$$

Show (36). If $A \geq B$, then $A = A \cup A \geq B \cup A$ by additivity, and $A \equiv A \cup B$ by monotonicity. Turn to (37). For all $x \in B$, $A \geq B \geq x$ implies $A \geq x$. Conversely, if $B = \{x_1, \ldots, x_n\}$, and $A \geq x_i$ for all $i = 1, \ldots, n$, then by (36),

$$
A \equiv A \cup x_1 \equiv A \cup x_1 \cup x_2 \equiv \ldots \equiv A \cup B.
$$

Turn to (38). If $A \cup D \supseteq D$ does not hold, then $A \cup D \equiv D$ by monotonicity, and hence $A \cup B \cup D \equiv B \cup D$ by additivity. By contradiction, (38) is true.

Proof of Theorem 9

Let $\mathcal{F} = \mathcal{F}(\geq)$ be the focus of $\geq$.

Take any $y \in Z$ and let

$$
\mathcal{X}(y, \geq) = \{ A \in \mathcal{M} : A \cup y \gg A \}.
$$

Let $F \in \mathcal{F}$ be focal for $y$. Then $F \cup y \gg F$ and hence, $F \in \mathcal{X}(y, \geq)$. Take any $A \ni F$ and $x \in A \setminus F$. As $F$ is focal for $y$, then $F \cup x \geq F \cup y$. By additivity,

$$
A = F \cup x \cup A \equiv F \cup y \cup A = A \cup y.
$$

Thus $A \not\in \mathcal{X}(y, \geq)$, and $F$ is maximal in $\mathcal{X}(y, \geq)$.

Let $F$ be maximal in $\mathcal{X}(y, \geq)$. Then $F \cup y \gg F$, but $F \cup x \cup y \equiv F \cup x$ for all $x \not\in F$. As $\geq$ is monotonic, then $F \cup x \geq F \cup x \cup y \equiv F \cup y$. Thus $F$ is focal for $y$.

Suppose that $\geq$ is induced by a weak order $\succeq$. Let

$$
\mathcal{X}(y, \succeq) = \{ A \in \mathcal{M} : A \cup y \not\succeq A \}.
$$

Let $F \in \mathcal{F}$ be focal for $y$. Take any $A \ni F$. As $F$ is maximal in $\mathcal{X}(y, \succeq)$, then $A \cup y \equiv A$. Thus $A \cup y \sim A$ and $A \not\in \mathcal{X}(y, \succeq)$. As $F$ is focal for $y$, then $F \cup y \gg F$. Thus there is $D \in \mathcal{M}$ such that $F \cup D \in \mathcal{X}(y, \succeq)$. If $F \cup D \not\supseteq F$, then $F \cup D \not\in \mathcal{X}(y, \succeq)$. Thus $F \cup D = F$ and hence, $F \in \mathcal{X}(y, \succeq)$ is a maximal element in the family $\mathcal{X}(y, \succeq)$.

Suppose that $F$ is maximal in $\mathcal{X}(y, \succeq)$. As $F \cup y \not\succeq F$, then $F \cup y \gg F$. Take any $x \not\in F$ and $D \in \mathcal{M}$. Then

$$
(F \cup x) \cup D \sim (F \cup x) \cup D \cup y = (F \cup x) \cup D \cup (F \cup y).
$$

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By definition of induced dominance, \( F \cup x \geq F \cup y \). Thus \( F \) is focal for \( y \).

Suppose that \( \geq \) is represented by a state space \( S \subset R \). Let

\[
\mathcal{X}(y, S) = \{ L'(y, R) : R \in S \}
\]

where \( L'(y, R) = \{ z \in Z : yPz \} \).

Let \( F \in \mathcal{F} \) be focal for \( y \). If \( FRy \) for all \( R \in S \), then \( F \cup y \cong F \). As \( F \cup y \gg F \), then \( F \subset L'(y, R) \) for some \( R \in S \). Wlog, \( L'(y, R) \) is a maximal element in \( \mathcal{X}(y, S) \) such that \( F \subset L'(y, R) \). If \( F \neq L'(y, R) \), then \( F \cup x \subset L'(y, R) \) for some \( x \in L'(y, R) \setminus F \). Then \( F \cup x \geq F \cup y \) does not hold and hence, \( F \) is not focal for \( y \). Thus \( F = L'(y, R) \) is maximal in \( \mathcal{X}(y, S) \).

Suppose that \( F \) is maximal in \( \mathcal{X}(y, S) \). Take \( R \in S \) such that \( F = L'(y, R) \). Then \( F \cup y \gg F \) because \( yPz \) for all \( z \in F \). Take any \( x \notin F \). If \( F \cup x \subset L'(y, R') \) for some \( R' \in S \), then \( F \) is not maximal in \( \mathcal{X}(y, S) \). Thus \( F \cup x \not\subset L'(y, R') \) for all \( R' \in S \). Thus \( F \cup x \gg F \cup y \) for all \( R' \in S \) and hence, \( F \cup x \geq F \cup y \). Thus \( F \) is focal for \( y \).

**Proof of Theorem 3**

The proof of Theorem 3 requires some preliminary lemmas.

Say that a menu \( C \in \mathcal{M} \) is central if for all \( y \notin C \), \( C \cup y \gg C \).

**Lemma A.1.** A menu \( C \in \mathcal{M} \) is central if and only if \( C \in \pi(\mathcal{F}) \).

**Proof.** Take any central \( C \in \mathcal{M} \) and \( y \notin C \). Then \( C \cup y \gg C \). Thus \( C \in \mathcal{X}(y, \geq) \).

Let \( F_y \) be a maximal menu in \( \mathcal{X}(y, \geq) \) such that \( C \subset F_y \). By Theorem 9, \( F_y \) is focal. Thus \( C = \cap_{y \notin C} F_y \) and hence, \( C \in \pi(\mathcal{F}) \).

Take any \( C \in \pi(\mathcal{F}) \) and \( x \notin C \). Then there is focal \( F \supset C \) such that \( x \notin F \). As \( F \) is focal for some \( y \), then \( F \cup x \geq F \cup y \gg F \). By (38),

\[
F \cup C \cup x \gg F \cup C \quad \Rightarrow \quad C \cup x \gg C.
\]

Thus \( C \) is central. \( \square \)

Recall that for any \( C \in \mathcal{M} \), \( \omega_C : Z \to \{0, 1\} \) is the Boolean function such that \( \omega_C(z) = 1 \) for all \( z \notin C \), and \( \omega_C(z) = 0 \) for all \( z \in C \).

**Lemma A.2.** For any state space \( S \subset R \) and menus \( A, B \in \mathcal{M} \),

\[
ARB \text{ for all } R \in S \quad \iff \quad \omega_C(A) \geq \omega_C(B) \text{ for all } C \in \mathcal{L}(S). \tag{39}
\]

**Proof.** Take any \( S \subset R \) and \( A, B \in \mathcal{M} \). Consider two possible cases.

**Case 1.** \( ARB \text{ for all } R \in S \). Take any \( C \in \mathcal{L}(S) \). Then

\[
C = \{ z \in Z : xR^*z \} \quad \text{for some } R^* \in S, x \in Z.
\]

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If \( \omega_C(B) = 0 \), then \( \omega_C(A) \geq \omega_C(B) \). Suppose that \( \omega_C(B) = 1 \), but \( \omega_C(A) = 0 \). Then there is \( y \in B \setminus C \), but \( A \subset C \). Thus \( yP^*z \) for all \( z \in A \) and \( AR^*B \) does not hold. Thus \( \omega_C(B) = 1 \) implies that \( \omega_C(A) = 1 \geq \omega_C(B) \).

**Case 2.** \( AR^*B \) does not hold for some \( R^* \in S \). Then there is \( x \in B \) such that \( xP^*z \) for all \( z \in A \). Take \( y \in R^*(A) \) to be some maximal element of \( R^* \) in \( A \). Take the lower contour set \( C = \{ z \in Z : yR^*z \} \in \mathcal{L}(S) \). Then \( \omega_C(B) = 1 \) because \( x \notin C \), but \( \omega_C(A) = 0 \) because \( A \subset C \).

These two cases imply the equivalence (39).

**□**

Turn to Theorem 3.

Let \( \geq \) be a dominance relation, and \( F \) its focus. Take any state space \( S \subset \mathcal{R} \). Consider two cases.

(I) \( F \subset \mathcal{L}(S) \subset \pi(F) \). I claim that for all \( A, B \in \mathcal{M} \),

\[
A \geq B \iff \omega_C(A) \geq \omega_C(B) \text{ for all } C \in \mathcal{L}(S).
\]

(40)

Take any \( A, B \in \mathcal{M} \) such that \( A \geq B \). Take any \( C \in \mathcal{L}(S) \). If \( \omega_C(A) = 1 \), then \( \omega_C(A) \geq \omega_C(B) \). Let \( \omega_C(A) = 0 \). Then \( A \subset C \). As \( A \geq B \), then \( C \geq B \). Take any \( x \in B \). By (36), \( C \geq C \cup B \geq C \cup x \). By Lemma A.1, \( C \in \pi(F) \) is central. As \( C \geq C \cup x \), then \( x \in C \). Thus \( B \subset C \), and \( \omega_C(B) = 0 \leq \omega_C(A) \).

Take any \( A, B \in \mathcal{M} \) such that \( \omega_C(A) \geq \omega_C(B) \) for all \( C \in \mathcal{L}(S) \). Suppose that \( A \cup y \gg A \) for some \( y \in B \). Then \( A \in \mathcal{X}(y, \geq) \). Take a maximal element \( F \in \mathcal{X}(y, \geq) \) such that \( F \supset A \). By Theorem 9, \( F \) is focal for \( y \). As \( F \supset A \), then \( \omega_F(A) = 0 \). As \( \mathcal{L}(S) \supset F \), then \( \omega_F(B) \leq \omega_F(A) = 0 \). Thus \( B \subset F \). However, \( y \in B \setminus F \) because \( F \cup y \gg F \). This contradiction shows that \( A \cup y \equiv A \). As \( y \in B \) is arbitrary, then by (37), \( A \geq B \). Therefore, (40) holds.

(II) (40) holds. Take any menu \( F \in F \) that is focal for \( y \in Z \). As \( F \cup y \gg F \), then by (40), \( \omega_C(y) > \omega_C(F) \) for some \( C \in \mathcal{L}(S) \). Thus \( \omega_C(y) = 1 \) and \( \omega_C(F) = 0 \), that is, \( y \notin C \) and \( F \subset C \). Suppose that \( x \in C \setminus F \). Then \( \omega_C(F \cup x) = 0 < \omega_C(y) \). By (40), \( F \cup x \gg F \cup y \) does not hold, and \( F \) is not focal for \( y \). This contradiction implies that \( C \setminus F \) is empty. Thus \( C = F \) and \( F \subset \mathcal{L}(S) \).

Take any \( C \in \mathcal{L}(S) \). Then for all \( x \notin C \), \( \omega_C(C) = 0 < 1 = \omega_C(C \cup x) \) and by (40), \( C \cup x \gg C \). Thus \( C \) is central, and \( \mathcal{L}(S) \subset \pi(F) \).

The equivalence of cases (I) and (II) implies that the set inclusions \( F \subset \mathcal{L}(S) \subset \pi(F) \) are equivalent to representation (40). By Lemma A.2, \( S \) represents \( \geq \) if and only if \( F \subset \mathcal{L}(S) \subset \pi(F) \).
Proof of Theorem 10.

No Compromise is equivalent to the completeness of the revealed dominance $\geq$. Indeed, if $\geq$ is complete, then for all $A, B \in \mathcal{M}$, $A \geq B$ or $B \geq A$ imply $A \sim A \cup B$ or $B \sim A \cup B$ respectively. Conversely, suppose that $\succeq$ satisfies No Compromise, but $\geq$ is not complete. Then there are $A, B, D, E \in \mathcal{M}$ such that $A \cup D \not\sim A \cup B \cup D$ and $B \cup E \not\sim A \cup B \cup E$. As

$$A \cup B \cup D = (A \cup D) \cup (A \cup B)$$

then No Compromise implies that $A \cup B \cup D \sim A \cup B$. Similarly,

$$A \cup B \cup E = (B \cup E) \cup (A \cup B)$$

implies $A \cup B \cup E \sim A \cup B$. No Compromise implies that

$$A \cup B \cup D \cup E = (A \cup B \cup D) \cup (A \cup B \cup E) \sim A \cup B \cup D \sim A \cup B \cup E.$$ 

Thus $A \cup B \cup D \cup E \not\sim A \cup D$ and $A \cup B \cup D \cup E \not\sim B \cup E$, which contradicts No Compromise. Thus $\geq$ is complete.

Therefore, any weak order that satisfies No Compromise can be represented by (22). Take $u$ and $v$ that represent $\succeq$ and $\geq$ on $Z$. As $\geq$ is monotonic and additive, then it is strategically rational: for all $A, B \in \mathcal{M},$

$$A \geq B \quad \Rightarrow \quad A = A \cup A \geq A \cup B \quad \Rightarrow \quad A \not\sim A \cup B.$$ 

As $\geq$ is complete, then

$$A \geq B \quad \Leftrightarrow \quad v(A) \geq v(B).$$

Then for all $x, z \in Z$, $v(x) = v(y)$ implies that $x \succeq y$ and hence, $u(x) = u(y)$.

Conversely, suppose that $\mathcal{F}$ is a chain. By Theorem 11, $\succeq$ is represented by $\phi(\omega_F)$ for some $\phi : \omega_F \circ Z \rightarrow \mathbb{R}$. As $\succeq$ is monotonic, then $\phi$ is monotonic and hence, increasing on the range of $\omega_F$. For all $A, B \in \mathcal{M}$, $\omega_F(A) \geq \omega_F(B)$ implies $A \succeq B$. Suppose $\omega_F(A) > \omega_F(B)$. Then there is focal $F \in \mathcal{F}$ such that

$$A \succeq B \quad \Rightarrow \quad \omega_F(A) \geq \omega_F(B) \quad \Rightarrow \quad \omega_F(A) = \omega_F(A \cup B) \quad \Rightarrow \quad A \sim A \cup B,$$

and $\succeq$ is strategically rational.

Proof of Theorem 11

Take any preference tuple $(U, \succeq, \geq, \mathcal{F})$ and a set $\Omega \subset \mathbb{R}^Z$.

Let $S = \mathcal{R}_\Omega = \{R_\omega : \omega \in \Omega\}$. Obviously, $\mathcal{L}(S) = \mathcal{L}(\Omega)$ and $\succeq$ is represented by $S$ if and only if it is represented by $\Omega$. Thus the equivalence of (i) and (ii) in Theorem 11 is established by Theorem 3.
Recall that $\Omega(A)$ denotes the vector of the values $\omega(A)$ for $\omega \in \Omega$. In the above notation, $\Omega$ represents $\succeq$ if and only if for all $A, B \in \mathcal{M}$,

$$A \succeq B \iff \Omega(A) \succeq \Omega(B). \quad (41)$$

Show that (iii) implies (ii). Suppose that $\succeq$ is represented by (26). Take any $A, B \in \mathcal{M}$ such that $A \succeq B$. By (36), $A \succeq A \cup B$. If $\Omega(A \cup B) = \Omega(A)$, then $\Omega(A) \succeq \Omega(B)$. Suppose that $\Omega(A \cup B) > \Omega(A)$. As $\phi$ is coherent, then there is $D \in \mathcal{M}$ such that

$$\phi(\Omega(A \cup B) \vee \Omega(D)) \neq \phi(\Omega(A) \vee \Omega(D)).$$

By (26), $A \cup D \not\succeq A \cup B \cup D$, which contradicts $A \succeq B$.

Take any $A, B \in \mathcal{M}$ such that $\Omega(A) \succeq \Omega(B)$. Then $\Omega(A) = \Omega(A \cup B)$. For all $D \in \mathcal{M}$, $\Omega(A \cup D) = \Omega(A \cup B \cup D)$ and hence, $A \cup D \sim A \cup B \cup D$. Thus $A \succeq B$.

Show that (ii) implies (iii). Suppose that (41) holds. As $\succeq$ is complete and transitive on the finite $\mathcal{M}$, then $\succeq$ is represented by some utility function $U : \mathcal{M} \to \mathbb{R}$. For any $a \in \Omega \circ \mathcal{M}$, take $A \in \mathcal{M}$ such that $a = \Omega(A)$, and let

$$\phi(a) = U(A).$$

The function $\phi$ is well-defined: if $a = \Omega(A) = \Omega(B)$ for another menu $B \in \mathcal{M}$, then by (41), $A \succeq B$ and $B \succeq A$, which implies $A \sim A \cup B \sim B$ and $U(A) = U(B)$. Show that $\phi$ is coherent. Take any $a, b \in \Omega \circ \mathcal{M}$ such that $a \neq b$. Take $A, B \in \mathcal{M}$ such that $\Omega(A) = a$ and $\Omega(B) = b$. Then $\Omega(A \cup B) > \Omega(B)$. By (41), $A \cup B \gg B$. Thus $A \cup C \not\sim A \cup B \cup C$ for some $C \in \mathcal{M}$. Take $c = \Omega(C)$. Then

$$\phi(a \lor (b \lor c)) = \phi(\Omega(A \cup B \cup C)) \neq \phi(\Omega(B \cup C)) = \phi(b \lor (b \lor c)).$$

Thus $\phi$ is coherent and by construction, $\phi(\Omega(A)) = U(A)$ represents $\succeq$ for all $A \in \mathcal{M}$. Any other representation $U^*(A) = \phi^*(\Omega(A))$ must be a strictly increasing transformation of $U$. Thus $\phi^*$ is a strictly increasing transformation of $\phi$.

If $\phi$ is monotonic, then for all $A, B \in \mathcal{M}$, $A \cup B \succeq A$ because $\Omega(A \cup B) \succeq \Omega(A)$. Conversely, assume that $\succeq$ is monotonic. Take any $\Omega(A), \Omega(B) \in \Omega \circ \mathcal{M}$ such that $\Omega(A) \succeq \Omega(B)$. Then $\Omega(A \cup B) = \Omega(A)$. Therefore, $\phi(\Omega(A)) = \phi(\Omega(A \cup B)) \geq \phi(\Omega(B))$.

Suppose that $\phi$ is submodular. Take any $A, B \in \mathcal{M}$ such that $A \sim A \cup B$. Then $\Omega(A \cup B) \succeq \Omega(A)$ and $\phi(\Omega(A)) = \phi(\Omega(A \cup B))$. As $\phi$ is submodular, then $\Omega(A) = \Omega(A \cup B)$. Thus for all $D \in \mathcal{M}$, $\Omega(A \cup D) = \Omega(A \cup B \cup D)$ and by (41), $A \cup D \sim A \cup B \cup D$. Therefore, $\succeq$ is submodular.

Suppose that $\succeq$ is submodular. Take any $A, B \in \mathcal{M}$ such that $\Omega(A) > \Omega(B)$. By (41), $A \gg B$, that is, $A \cup B \cup D \not\sim B \cup D$ for some $D \in \mathcal{M}$. By submodularity, $A \cup B \not\sim B$. Thus $\phi(\Omega(A)) = \phi(\Omega(A \cup B)) \neq \phi(\Omega(B))$, and $\phi$ is submodular.
Proofs of Theorem 13

Suppose that $C$ is essential for some dominance $\succeq$. Show that $\succeq$ satisfies (32). Indeed, if $A \succeq B$, then $A \succeq z$ for any $z \in B \setminus A$ and hence, $A \cup z \equi A$ and $z \not\in C(A \cup z)$. Conversely, suppose that $z \not\in C(A \cup z)$ for all $z \in B \setminus A = \{z_1, \ldots, z_n\}$. Then $A \simeq A \cup z_i$ for all $i = 1, \ldots, n$, and by additivity

$$A \simeq A \cup z_1 \simeq A \cup z_1 \cup z_2 \simeq \ldots \simeq A \cup B.$$

Thus $A \simeq A \cup B \succeq B$ and $A \succeq B$.

Suppose that $C$ satisfies Chernoff and Substitution. Let $\succeq$ satisfy (32). Show that $\succeq$ is a dominance relation. Monotonicity is vacuous. Additivity is implied by Chernoff. Show that $\succeq$ is transitive. Let $A \succeq B \succeq C$ and $z \in C \setminus A$. Assume that $z \not\in C(A \cup B)$. If $z \not\in B$, then this assumption contradicts $A \succeq B$. Thus $z \not\in B$. Write $B \setminus A = \{y_1, \ldots, y_n\}$. Note that $y_i \not\in C(A \cup y_i)$ for all $i$. By Substitution and Chernoff,

$$z \in C(A \cup z) \Rightarrow z \in C(A \cup z \cup y_1) \Rightarrow z \in C(A \cup z \cup y_1 \cup y_2) \Rightarrow \ldots$$

$$z \in C(A \cup B \cup z) \Rightarrow z \in C(B \cup z).$$

It contradicts $B \succeq C$. By contradiction, $z \not\in C(A \cup z)$ and $A \succeq C$.

Let $C^*$ be the essential function for $\succeq$. Then for all $A \in \mathcal{M}$ and $z \in A$,

$$z \not\in C(A) \iff A \setminus z \succeq A \iff z \not\in C^*(A)$$

by (32) and (31). Thus $C = C^*$ and hence, $C$ is essential for $\succeq$.

Proofs of Theorem 14

Suppose that $C$ is path independent. Then $C$ satisfies Chernoff. By Theorem 13, $C$ is essential for the dominance $\succeq$ that is represented by (32). Show that $\succeq$ is weakly total. Take any $A, B \in \mathcal{M}$ such that $A \equi B$. By additivity, $A \equiv A \cup B \equiv B$. Take any $z \in C(A \cup B)$. If $z \not\in A$, then $z \in C(A \cup z)$ and hence, $A \succeq A \cup B$ does not hold. Thus $z \in A$. Analogously, $z \in B$. Thus $C(A \cup B) \subset A \cap B$, which is not empty. Take any $x \in B \setminus A$. By path independence,

$$C((A \cap B) \cup x) = C((A \cap B) \cup x \cup C(A \cup B)) = C(A \cup B) \subset A \cap B.$$

By (32), $A \cap B \succeq B$ and hence, $A \cap B \equiv B$. Thus $\succeq$ is weakly total.

Suppose that $C$ is essential for a weakly total dominance $\succeq$. Then for all $A \in \mathcal{M}$,

$$A \equiv C(A) = \bigcap_{z \in A \setminus C(A)} A \setminus z$$

(42)

because $A \equiv A \setminus z$ for all $z \in A \setminus C(A)$. Show that $C$ is path independent. Take any $A, B \in \mathcal{M}$. Let $D = C(A) \cup B$. Show that $C(A \cup B) \subset C(D)$. Take any
If $z \in A$, then by Chernoff, $z \in C(A)$ and hence, $z \in C(D)$. If $z \in B$, then by Chernoff, $z \in C(D)$.

Show that $C(D) \subset C(A \cup B)$. Take any $z \not\in C(A \cup B)$. By (42), $A \equiv C(A)$. By additivity, $A \cup B \equiv C(A) \cup B = D$. By weak totality, $(A \cup B) \setminus z \equiv A \cup B \equiv D$ implies that

$$D \equiv D \cap ((A \cup B) \setminus z) = D \setminus z.$$

Thus $z \not\in C(D)$.

References


