FRAMING IN EXPECTED UTILITY AND MULTIPLE PRIORS MODELS

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Abstract

I extend expected utility and multiple priors models to accommodate several framing effects, such as partition dependence, comparative ignorance, and preference reversals. These patterns can be formally revealed through non-monotonic and intransitive choices among state-contingent prospects in Anscombe-Aumann’s standard framework. By relaxing monotonicity and transitivity, I characterize bi-partitional representations where any state-contingent prospects \( f \) and \( g \) are compared via probabilistic beliefs or sets of beliefs that depend on the partitions generated by \( f \) and \( g \). For any such partitions \( \pi \) and \( \tau \), the corresponding beliefs \( p(\pi, \tau) \) and sets \( M(\pi, \tau) \) are determined uniquely on the partition \( \pi \), but need not be preserved when \( \pi \) or \( \tau \) vary. Additional structure for the functions \( p(\cdot) \) and \( M(\cdot) \) is obtained from various combinations of transitivity and monotonicity properties. One special case (cross-partitional expected utility) can be combined with Ahn and Ergin’s (2010) results to derive Tversky and Koehler’s support theory for the belief function \( p(\cdot) \).

1 Introduction

Empirical research provides many framing patterns that violate monotonicity and transitivity of preferences. To illustrate, recall several experiments for choices under uncertainty.

Tversky and Kahneman (TK) [14] observe representativeness and availability heuristics where subjective confidence in a proposition \( A \) can be often increased

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by additional constraints that are (i) salient or representative for $A$ or (ii) make it cognitively easier to produce examples that support $A$. For example, a majority of TK’s subjects choose to bet on a random sequence $GRGRRR$ rather than on $RGRRR$ when the outcome $G$ is known to be twice as likely as its alternative $R$. Here the longer sequence $GRGRRR$ can appear more representative than $RGRRR$ because it increases the proportion of the more likely outcomes $G$. The most famous example of this sort is the “Linda effect”, but it does not involve monetary bets because Linda is fictional.\(^1\)

To illustrate the availability heuristics, TK observe that most subjects find it more likely for a seven-letter English word to have the form ----ing rather than -----n- presumably because the former sequence has more readily available examples than the latter.

Similarly, availability heuristics can motivate partition-dependent betting behaviors where two separate tickets $t(A)$ and $t(B)$ that pay some monetary prize $Z > 0$ on the disjoint events $A$ and $B$ are typically valued more highly than the single equivalent ticket $t(A \cup B)$. Sonnemann, Camerer, Fox, and Langer \cite{12} provide empirical evidence: they set up experimental markets for contingent assets

$$t(A) = \begin{cases} 
$1 & \text{if } x \in A \\
$0 & \text{if } x \notin A
\end{cases}$$

where $x$ is some random variable (like the temperature in Muenster, the DAX index closing value, or the number of goals in a Bundesliga game day), and $A$ is some interval of the real line (e.g. $A = [0, 20]$ for the daily number of Bundesliga goals). In market $M_1$, $t(A)$ was traded together with two other assets $t(B_1)$ and $t(C_1)$ such that $B_1$ and $C_1$ partitioned the event $A$. In market $M_2$, $t(A)$ was traded together with $t(B_2)$ and $t(C_2)$ such that $B_2$ and $C_2$ partitioned the complement of $A$. The former treatment produced substantially higher prices (by about 20-25 cents) for $t(A)$ than the latter. The gap in the price occurred even though participants in each market were informed about the other market as well. Prior to Sonnemann et al., partition-dependence has been identified in various contexts with verbally reported beliefs (e.g. Fischhoff, Slovic, and Lichtenstein \cite{6}, Fox and Birke \cite{7} and others).

Another broad type of framing is the comparative ignorance effect observed for ambiguity averse subjects. Based on their experimental data, Fox and Tversky \cite{8} assert that “ambiguity aversion will be present when subjects evaluate clear

\(^1\)She is portrayed as follows.

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social injustice, and also participated in anti-nuclear demonstrations.

TK observe that 80% of subjects believe that Linda is more likely to be a feminist bank teller rather than just a bank teller.
and vague prospects jointly, but it will greatly diminish or disappear when they evaluate each prospect in isolation.” In one experiment, Fox and Tversky asked UC Berkeley students to evaluate tickets $t_I$ and $t_{SF}$ paying $100 if the afternoon temperature on a given day would exceed 60F in Istanbul and in San Francisco respectively. The average cash equivalent for the ticket $t_I$ was $38 when $t_I$ was priced on its own, but only $24 when $t_I$ was priced together with $t_{SF}$. Thus the subjective value of the prospect $t_I$ was reduced in the context of the bet $t_{SF}$ on the more familiar event even though this context did not add any information to the subjects’ prior knowledge about the Istanbul climate. Fox and Tversky conclude that subjects’ ambiguity aversion is driven by the “feeling of incompetence” that can be context dependent.

In addition to non-monotonic choice patterns, framing can generate intransitive preference reversals similar to those that Slovic and Lichtenstein [11] describe for risky gambles (see also Tversky, Slovic, and Kahneman [16]). Formally, a preference reversal occurs when the direct choice between uncertain prospects $f$ and $g$ is inconsistent with the cash equivalents $C_f$ and $C_g$ that are elicited separately for $f$ and $g$:

$$C_f \sim f \succ g \sim C_g > C_f.$$  \hspace{1cm} (1)

Clearly, this ranking is intransitive whenever more money is better than less.

For example, let $f$ and $g$ be monetary bets that pay $100 contingent on a random English word having the form ‘-----n-’ and ‘----ing’ respectively. When the bets are evaluated separately via cash equivalents, $C_f < C_g$ is plausible for some agents because of the availability heuristic. However, the strict ranking $f \succ g$ may still result from a direct comparison between $f$ and $g$, which can make it clearer that $f$ dominates $g$.

Preference reversals (1) can also result from comparative ignorance if $g$ is a bet on a more ambiguous event than $f$. Trautmann, Vieider, and Wakker [13] observe preference reversals empirically for bets on ambiguous events, though their data does not fully agree with Fox and Tversky’s predictions. In fact, Trautman et al. find that the preference reversals (1) are more common when $f$ is a bet on a more ambiguous event than $g$.

**Representations with Framing Effects**

This paper extends the standard expected utility (EU) and maxmin expected utility (MEU, Gilboa and Schmeidler [9]) models accommodate both non-monotonic and intransitive patterns within the regular decision framework of Anscombe and Aumann [2].

Consider preferences $\succeq$ over uncertain prospects (acts) that map a finite state space $S$ into a space of lotteries $X$. Given any act $f$ and probability distribution $p$ on $S$, let $f(p)$ be the induced lottery $\sum_{s \in S} p(s) f(s)$.
Consider a very general representation

$$f \succeq g \iff u(f(b(f, g))) \succeq u(b(g, f)), \quad (2)$$

where $u : X \rightarrow \mathbb{R}$ is some expected utility index, and the beliefs $b(f, g)$ and $b(g, f)$ can be framed by the acts $f$ and $g$. By making the beliefs depend on both evaluated prospects, one can accommodate various framing patterns together with ambiguity aversion. A simple proposition (Theorem 1 below) asserts that representation (2) must exist for any complete preference $\succeq$ that complies with expected utility over lotteries, and satisfies a weak form of monotonicity.

However, the absolute freedom in the assignment of the beliefs $b(f, g)$ makes it impossible to identify any of them uniquely. Moreover, representation (2) guarantees completeness, but can violate all other conditions of the EU and MEU models: transitivity, continuity, monotonicity, Certainty Independence, and a fortiori, Independence.

In this paper, I propose several refinements of the universal model (2) where the framing effects of the feasible prospects $f$ and $g$ are determined by the partitions $\pi_f$ and $\pi_g$ that $f$ and $g$ generate on the state space. The most important representations are

$$f \succeq g \iff u(f(p(\pi_f, \pi_g))) \succeq u(g(p(\pi_g, \pi_f))), \quad (3)$$

$$f \succeq g \iff \min_{q \in M(\pi_f, \pi_g)} u(f(q)) \geq \min_{q \in M(\pi_g, \pi_f)} u(g(q)), \quad (4)$$

where the beliefs $p(\pi_f, \pi_g)$ and sets of beliefs $M(\pi_f, \pi_g)$ are framed by the generated partitions $\pi_f$ and $\pi_g$. Representations (3) and (4) are called bi-partitional expected utility (BPEU) and bi-partitional maxmin expected utility (BPMEU) respectively. The added partitional structure

- allows to identify the beliefs $p(\pi, \tau)$ and sets $M(\pi, \tau)$ uniquely on the partition $\pi$ (which is sufficient to compute all evaluations in (3) and (4)),
- implies Certainty Independence as is together with weak versions of transitivity, continuity, and uncertainty aversion.

My main results (Theorems 2 and 3 below) characterize BPEU and BPMEU in terms of these conditions. Note that BPEU is derived as a special case of BPMEU: in the presence of framing effects, the multiple priors model paves the way to expected utility representations!

Moreover, BPEU and BPMEU models allow to interpret regular transitivity and monotonicity in terms of simple additional properties of the functions $p(\cdot)$ and $M(\cdot)$. For all partitions $\pi$, $\tau$,

(i) transitivity is equivalent to the univariate structure $p(\pi, \tau) = p(\pi, \pi)$ and $M(\pi, \tau) = M(\pi, \pi)$,
(ii) monotonicity is equivalent to the symmetry \( p(\pi, \tau) = p(\tau, \pi) \) and \( M(\pi, \tau) = M(\tau, \pi) \),

(iii) a weaker form of transitivity (mixture transitivity) is equivalent to the cross-partitional structure \( p(\pi, \tau) = p(\pi, \pi \vee \tau) \) and \( M(\pi, \tau) = M(\pi, \pi \vee \tau) \),

(iv) the combination of monotonicity and mixture transitivity is equivalent to the univariate cross-partitional structure

\[
p(\pi, \tau) = p(\pi \vee \tau, \pi \vee \tau)
\] (5)

and \( M(\pi, \tau) = M(\pi \vee \tau, \pi \vee \tau) \).

As customary, the cross-partitions \( \pi \vee \tau \) are defined as the minimal partitions that are finer than both \( \pi \) and \( \tau \).

1.1 Unframable Partitions and Source Dependence

Chew and Sagi [4] model source dependence where preferences comply with expected utility (or more generally, probabilistic sophistication) when acts \( f \) and \( g \) share one source of uncertainty, but become ambiguity averse otherwise.\(^2\) Such source dependence is easily obtained within the BPMEU model. For example, let

\[ S = \{H, T\} \times \{R, B\} = \{HR, HB, TR, TB\} \]

be a state space that combines flipping a coin (\( H \) or \( T \)) and drawing a colored ball from an urn (\( R \) or \( B \)). Let \( \pi_1 = \{H, T\} \) and \( \pi_2 = \{R, B\} \) be the partitions generated by the two distinct sources of uncertainty. If \( \pi \vee \tau = \pi_1 \) or \( \pi \vee \tau = \pi_2 \), let \( M(\pi, \tau) = \{p_0\} \), where \( p_0 \) assigns equal probabilities \( \frac{1}{4} \) to all states in \( S \). If \( \pi \vee \tau \neq \pi_1 \) and \( \pi \vee \tau \neq \pi_2 \), let \( M(\pi, \tau) \) be the set of all probability measure \( q \) on \( S \) such that \( q(s) \geq \frac{1}{6} \) for all \( s \in S \). Then \( \succeq \) must comply with expected utility across \( \pi_1 \) measurable prospects (bets on coin flips) and across \( \pi_2 \) measurable prospects (bets on the urn drawings). However, if two sources are combined, then \( \succeq \) will exhibit some ambiguity aversion for both kinds of bets.\(^3\)

Moreover, the BPMEU and BPEU models allow stronger forms of source dependence. Call a partition \( \pi^* \) unframable if there are representations (3) or (4) where all beliefs \( p(\pi, \tau) \) or respectively, all sets \( M(\pi, \tau) \) agree on \( \pi^* \). Unframable partitions can be defined explicitly in terms of preference \( \succeq \) in my model. Given two unframable partitions \( \pi^* \) and \( \tau^* \), the cross-partition \( \pi^* \vee \tau^* \) can be framable. Recall Tversky and Kahneman’s example where subjects are informed that the

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\(^2\) Chew and Sagi focus on more special type of framing and use primitives, axioms, and utility representations that are all distinct from mine.

\(^3\) If necessary, one can adjust the function \( M(\cdot) \) so that only one of the two sources becomes ambiguous when they are mixed.
outcome $G$ is twice as likely as its alternative $R$. Then the partitions $\{G,R\}$ can be unframable for each outcome realization, while the cross-partition of six copies of $\{G,R\}$ is framable if the DM prefers to bet on $GRGRRR$ rather than on $RGRRR$.

1.2 Partition Dependence and Support Theory

Ahn and Ergin’s [1] model of partition dependent expected utility (PDEU) has the cross-partitional structure (5) for preferences $\succeq^*$ over act-partition pairs $(f, \pi)$ such that $\pi$ can be finer than the partition $\pi_f$ generated by $f$. This setting allows to frame the comparison of $f$ and $g$ by exogenous partitions rather than just by events that can be defined in terms of the two prospects $f$ and $g$.

Any pairs $(f, \pi)$ and $(g, \tau)$ are compared via partition-dependent expected utility (PDEU)

$$(f, \pi) \succeq^*(g, \tau) \iff u(f(p(\pi \lor \tau))) \geq u(g(p(\pi \lor \tau))),$$

and the subjective beliefs $p(\pi)$ satisfy Tversky and Koehler’s [15] support theory: for any partition $\pi = \{E_1, \ldots, E_n\}$ and event $E_i \in \pi$,

$$p(E_i, \pi) = \frac{\nu(E_i)}{\sum_{j=1}^{n} \nu(E_j)}$$

for some non-negative support function $\nu$ such that $\sum_{E_j \in \pi} \nu(E_j) > 0$. In my framework, the additional structure (7) can be derived from Ahn and Ergin’s results (their Theorem 3) and their key axioms, the Sure-Thing Principle and Binary Bet Acyclicity.

2 Preliminaries

Consider a version of Anscombe–Aumann’s [2] decision framework. Let $S = \{s, \ldots\}$ be a finite state space. Call all of its subsets $E \subset S$ events.

Fix an arbitrary set $D$ of deterministic payoffs. Let $X = \{x, y, z, \ldots\}$ be the set of all lotteries—probability distributions on $D$ that have a finite support and are resolved after the state $s \in S$ is observed.

Let $\mathcal{H} = \{f, g, h \ldots\}$ be the set of all acts—functions $f : S \to X$. Interpret each act $f \in \mathcal{H}$ as a physical action that delivers lotteries $f(s)$ contingent on the state $s \in S$. Each lottery $x \in X$ is identified with the constant act $x \in \mathcal{H}$.

For any $f, g \in \mathcal{H}$ and $\alpha \in [0, 1]$, define the mixture $\alpha f + (1 - \alpha) g$ as

$$[\alpha f + (1 - \alpha) g](s) = \alpha f(s) + (1 - \alpha) g(s) \quad \text{for all } s \in S.$$
For any event \( E \subset S \), define the composite act \( fEg \) as

\[
[fEg](s) = \begin{cases} 
    f(s) & \text{if } s \in E \\
    g(s) & \text{if } s \notin E
\end{cases}
\]

for all \( s \in S \).

Consider a decision maker (DM) who has a weak preference \( \succeq \) over \( \mathcal{H} \) with asymmetric and symmetric parts \( \succ \) and \( \sim \) respectively. Assume throughout that

\begin{itemize}
  \item \( \succ \) is not empty, and
  \item \( \succeq \) is complete: for any \( f, g \in \mathcal{H} \), either \( f \succeq g \) or \( g \succeq f \) must hold.
\end{itemize}

Note that preferences \( \succeq \) are defined over Anscombe–Aumann’s acts \( f \in \mathcal{H} \) rather than over combinations of acts and partitions as in Ahn and Ergin [1].

It is often practical to describe events \( E \subset S \) parsimoniously without listing all states \( s \in E \). For example, propositions like “an English word ends with the letter g” in TK’s experiments are naturally formulated without listing all such words from a dictionary. In financial settings, investors can contemplate betting on events like “the stock market will crash by more than 10% next month” or “the interest rate will be raised at the next Fed meeting” without listing all conceivable realizations of stock prices or Fed announcements that belong to these events.

Thus, the comparison between any two acts \( f, g \in \mathcal{H} \) can be framed by parsimonious descriptions of these prospects. To accommodate such framing, consider

**Axiom 1 (Range Monotonicity (RM)).** For all \( f, g \in \mathcal{H} \), if \( f(s) \succeq (\succ)g(s') \) for all \( s, s' \in S \), then \( f \succeq (\succ)g \).

This condition is plausible for any descriptions of \( f \) and \( g \) that specify all possible outcomes \( f(s) \) and \( g(s') \), but not necessarily list all states \( s \) and \( s' \). Indeed, the rankings \( f \succeq (\succ)g \) are compelling when the DM (strictly) prefers any possible outcome \( f(s) \) of the act \( f \) to any possible outcome \( g(s') \) of the act \( g \), and her risk attitude—preference over lotteries in \( X \)—is state-invariant and immune to framing.

Let \( \Delta = \{p, q, \ldots\} \) is the simplex of all probability distributions on \( S \).

Let \( \mathcal{U} \) be the set of all non-constant expected utility functions \( u : X \to \mathbb{R} \). A positive linear transformation (plt) of any expected utility function \( u \in \mathcal{U} \) has the form \( \alpha u + \beta \) for \( \alpha > 0 \) and \( \beta \in \mathbb{R} \).

Say that \( \succeq \) is regular if \( \succeq \) has an expected utility representation \( u \in \mathcal{U} \) over \( X \)—the set of all constant acts. It is straightforward to show that RM and regularity characterize a very broad model of framing.

**Theorem 1.** A regular preference \( \succeq \) satisfies RM iff it is represented by

\[
    f \succeq g \quad \Leftrightarrow \quad u(f(b(f, g))) \succeq u(g(b(g, f)))
\]

for some \( u \in \mathcal{U} \) and \( b : \mathcal{H} \times \mathcal{H} \to \Delta \). Moreover, \( u \) is unique up to a plt.
Proof. Obviously, (8) implies RM and regularity. Conversely, suppose that $\succeq$ is regular and RM holds. Let $u \in \mathcal{U}$ represent $\succeq$ on $X$. Take any $f, g \in \mathcal{H}$. Assume $f \succ g$. By RM, there are $s, s' \in S$ such that $f(s) \succ g(s')$. Let $p(s) = 1$ and $q(s') = 1$. Then (8) holds for $b(f, g) = p$ and $b(g, f) = q$. Assume $f \sim g$. By RM, there are $s, s' \in S$ such that $f(s) \succeq g(s)$ and $g(s') \succeq f(s')$. Then

$$\alpha(u(f(s)) - u(g(s))) = (1 - \alpha)(u(g(s')) - u(f(s')))$$

for some $\alpha \in [0, 1]$. Take $p \in \Delta$ such that $p(s) = \alpha$ and $p(s') = 1 - \alpha$. Then $u(f(p)) = u(g(p))$, and (8) holds for $b(f, g) = b(g, f) = p$. \qed

Theorem 1 asserts that for regular preferences, RM is necessary and sufficient for representation (8), where the beliefs $b(f, g)$ and $b(g, f)$ can be framed in an arbitrary way by the acts $f$ and $g$. Moreover, $b(f, g)$ need not equal $b(g, f)$.

Note that (8) is not a utility representation and can contradict all basic rationality postulates, except for completeness and regularity. To illustrate, recall several standard axioms.

(i) Transitivity: for all $f, g, h \in \mathcal{H}$, $f \succeq g \succeq h$ implies $f \succeq h$.

(ii) Monotonicity: for all $f, g \in \mathcal{H}$, if $f(s) \succeq g(s)$ for all $s \in S$, then $f \succeq g$.

(iii) Archimedean Axiom: for any $f, g, h \in \mathcal{H}$ such that $f \succ g$, there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)g \succ g$ and $f \succ \beta g + (1 - \beta)h$.

(iv) Independence Axiom: for all $f, g, h \in \mathcal{H}$ and $\alpha \in (0, 1]$,

$$f \succeq g \iff \alpha f + (1 - \alpha)h \succeq \alpha g + (1 - \alpha)h.$$ 

All of these conditions can be violated together by representation (8). To find such a counterexample, fix an index $u \in \mathcal{U}$, an outcome $z \in X$, and let $\mathcal{H}_z$ be the set of all acts $f \in \mathcal{H}$ such that $u(f(s)) > u(z) > u(f(s'))$ for some $s, s' \in S$. Take an arbitrary ranking $\succeq_z$ of $\mathcal{H}_z$ that violates transitivity, monotonicity, Archimedean, and Independence.\(^5\) Note that regularity and RM hold vacuously for $\succeq_z$ because $\mathcal{H}_z$ contains neither constant acts nor pairs of acts that can be compared by RM. Thus $\succeq_z$ can be extended to a complete preference $\succeq$ on $\mathcal{H}$ that satisfies regularity and RM. By Theorem 1, $\succeq$ has representation (8). Yet it violates transitivity, monotonicity, Archimedean, and Independence.

Consider a weaker version of Independence that Gilboa and Schmeidler [9] propose to accommodate ambiguity aversion.

\(^5\)To do so, take $f, g, h \in \mathcal{H}_z$ such that all of their mixtures belong to $\mathcal{H}_z$ as well. For example, fix $s, s' \in S$ such that $f(s) = g(s) = h(s) \succ z \succ f(s') = g(s') = h(s')$ and let $f, g, h$ differ on the rest of the state space $S$. 

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Axiom 2 (Certainty Independence (CI)). For all $\alpha \in (0, 1]$, $f, g \in \mathcal{H}$, and $x \in X$,
\[
f \succeq g \iff \alpha f + (1 - \alpha)x \succeq \alpha g + (1 - \alpha)x.
\]

In general, CI need not hold in model (8) because CI restricts the ranking of the subdomain $\mathcal{H}_x$, and (8) does not.

In this paper, I impose CI to refine the framing model (8) and obtain some additional structure for the subjective beliefs $b(f, g)$. Similarly, I relax the Archimedean Axiom.

Axiom 3 (Certainty Archimedean (CA)). For any $f, g \in \mathcal{H}$ and $x \in X$ such that $f \succ g$, there are $\alpha, \beta \in (0, 1)$ such that $f \succ \alpha g + (1 - \alpha)x$ and $\beta f + (1 - \beta)x \succ g$.

Both CI and CA rely only on mixtures of the form $\alpha f + (1 - \alpha)x$ for an arbitrary act $f$ and a constant $x$. These mixtures can alleviate concerns about framing because either of the acts $f$ and $\alpha f + (1 - \alpha)x$ can be described in terms of the other without specifying any additional details of the state space. For example, once $f$ is defined, the mixture $\alpha f + (1 - \alpha)x$ can be derived by replacing each possible outcome $y$ of the act $f$ with $z = \alpha y + (1 - \alpha)x$. The inverse translation is also simple: $f$ replaces each possible outcome $z$ of the mixture $\alpha f + (1 - \alpha)x$ with the uniquely defined lottery $y$ such that $z = \alpha y + (1 - \alpha)x$. Neither of these translations mentions any particular events or states of the world.

3 Main Models

Let $\Pi = \{\pi, \tau, \ldots\}$ be the set of all partitions of the state space $S$. Each partition $\pi \in \Pi$ is a collection $\{E_1, \ldots, E_k\}$ of non-empty disjoint events such that $S = E_1 \cup E_2 \cup \cdots \cup E_k$.

Take any $f \in F$, $x \in X$, $q \in \Delta$, and $\pi, \tau \in \Pi$. Then
- $f(q) = \sum_{s \in S} q(s)f(s)$ is the lottery induced by $f$ via $q$,
- $f^{-1}(x) = \{s \in S : f(s) = x\}$,
- the partition $\pi(f) \in \Pi$, or $\pi f$ for short, is generated by $f$ if it consists of all non-empty events $E$ such that $E = f^{-1}(y)$ for some $y \in X$,
- the cross-partition $\pi \vee \tau \in \Pi$ consists of all non-empty events $E = E \cap T$ such that $E \in \pi$ and $T \in \tau$,
- if $\pi \vee \tau = \pi$, say that $\pi$ is finer than (refines) $\tau$, and $\tau$ is coarser than $\pi$,
- $\Delta_\pi$ is the simplex of all distributions $p : \pi \to [0, 1]$ such that $\sum_{E \in \pi} p(E) = 1$,
- $q_\pi \in \Delta_\pi$ is the restriction of $q$ to $\pi$ such that $q_\pi(E) = q(E)$ for all $E \in \pi$.
• $\mathcal{H}_\tau$ is the set of all acts $h \in \mathcal{H}$ such that $\pi h = \tau$,

• $\mathcal{I} f = \mathcal{H}_{\pi f}$ is the set of all acts $h \in \mathcal{H}$ such that $\pi h = \pi f$.

Here the notation $\mathcal{I} f$ is motivated by the well-known fact that $h \in \mathcal{I} f$ if and only if $h$ is a composition $h = \theta \circ f$ for some injective\(^6\) mapping $\theta : X \to X$. Therefore, any act $h \in \mathcal{I} f$ can be described as a modification of the outcomes of $f$ without adding or omitting any relevant details about the state space. Note that $\mathcal{I} f = \mathcal{I} h$ for any $h \in \mathcal{I} f$ so the inverse translation from $h$ into $f$ is also possible.

### 3.1 Bi-Partitional Expected Utility (BPEU)

Suppose that the beliefs $b(f, g)$ in representation (8) can be framed by the generated partitions $\pi f$ and $\pi g$, but must be invariant of all other aspects of the feasible prospects $f$ and $g$. Then for all $f, g, h \in \mathcal{H}$, the following conditions should hold:

**B1:** if $h \in \mathcal{I} g$, then $b(f, g) = b(f, h)$,

**B2:** if $h \in \mathcal{I} f$, then $b(f, g) = b(h, g)$.

I show below (Section 4.5) that B1 and B2 imply CI, CA, and two other properties for the preference $\succeq$.

Write $f \succeq g$ if $f(s) \succeq g(s)$ for all $s \in S$.

**Axiom 4** (Partition-Preserving Transitivity (PPT)). For all $f, g \in \mathcal{H}$, $f' \in \mathcal{I} f$, and $g' \in \mathcal{I} g$,

$$f \succeq g' \succeq f' \succeq g \quad \text{or} \quad f \succeq f' \succeq g' \succeq g \quad \Rightarrow \quad f \succeq g.$$  \hfill (9)

Note that PPT must hold when preferences are monotonic and transitive, but does not assume either of these properties. Instead, PPT implies that $\succeq$ is both monotonic and transitive on any subdomain $\mathcal{I} f \subset \mathcal{H}$ of acts that generate the same partition $\pi f$. In this case, one can take $f' = g$, or $g' = g$, or both.

**Axiom 5** (Partition-Preserving Betweenness (PPB)). For all $\alpha \in [0, 1]$ and $f, g, h \in \mathcal{H}$ such that $g \in \mathcal{I} f$ and $\alpha f + (1 - \alpha)g \in \mathcal{I} f$,

$$f \succeq h \quad \text{and} \quad g \succeq h \quad \Rightarrow \quad \alpha f + (1 - \alpha)g \succeq h$$

$$h \succeq f \quad \text{and} \quad h \succeq g \quad \Rightarrow \quad h \succeq \alpha f + (1 - \alpha)g.$$  

\footnote{As customary, $\theta : X \to X$ is called injective if for all $x, y \in X$, $x \neq y$ implies $\theta(x) \neq \theta(y)$.}
For transitive preferences, PPB relaxes Dekel’s [5] Betweenness. Note that in the standard expected utility model, Independence is equivalent to the combination of CI and Betweenness. This decomposition into homotheticity and betweenness components is used by Burghart, Epper, Fehr [3] for preferences over lotteries. In my model, CI is preserved as is, but Betweenness must be restricted to prospects \( f, g, \alpha f + (1 - \alpha)g \) that generate the same partition.

**Theorem 2.** \( \succeq \) satisfies Axioms 1-5 iff it is represented by

\[
 f \succeq g \iff u(f(p(\pi f, \pi g))) \geq u(g(p(\pi g, \pi f))),
\]

where \( u \in \mathcal{U} \) and \( p : \Pi \times \Pi \to \Delta \).

Moreover, for any such preference \( \succeq \),

(i) \( u \) is unique up to a plt, and the restrictions \( p_\pi(\pi, \tau) \in \Delta_\pi \) are unique for all \( \pi, \tau \in \Pi \),

(ii) \( \succeq \) is monotonic iff \( \succeq \) is represented by (10) where \( p(\pi, \tau) = p(\tau, \pi) \) for all \( \pi, \tau \in \Pi \),

(iii) \( \succeq \) is transitive iff \( \succeq \) is represented by (10) where \( p(\pi, \tau) = p(\pi, \pi) \) for all \( \pi, \tau \in \Pi \),

(iv) \( \succeq \) is monotonic and transitive iff \( \succeq \) is represented by (10) where \( p(\pi, \tau) = p(\pi', \tau') \) for all \( \pi, \tau, \pi', \tau' \in \Pi \).

This result is derived in the appendix as a special case of Theorem 3 below.

The DM as portrayed by (10) compares any acts \( f \) and \( g \) via subjective beliefs \( p(\pi f, \pi g) \) and \( p(\pi g, \pi f) \) framed by the pair of partitions \( \pi = \pi f \) and \( \tau = \pi g \) generated by \( f \) and \( g \) respectively. Accordingly, (10) is called bi-partitional expected utility (BPEU) representation. This model violates monotonicity, transitivity, Archimedean, and Independence, but can be characterized via a list of weaker conditions, Axioms 1–5.

The uniqueness claim of Theorem 2 asserts that \( \succeq \) has two distinct representations (10) with components \((u, p)\) and \((u^*, p^*)\) where \( u, u^* \in \mathcal{U} \) and \( p, p^* : \Pi \times \Pi \to \Delta \) if and only if \( u^* \) is a plt of \( u \), and \( p_\pi^*(\pi, \tau) = p_\pi(\pi, \tau) \) for all \( \pi, \tau \in \Pi \). Therefore, the preference \( \succeq \) identifies the belief \( p(\pi, \tau) \) uniquely on the corresponding partition \( \pi \), but does not dictate how the restriction \( p_\pi(\pi, \tau) \) should be extended from \( \pi \) to a probability measure on the entire state space \( S \). Any such extension

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7In the current framework, Dekel’s Betweenness requires that for all \( \alpha \in (0, 1) \) and \( f, g, h \in \mathcal{H} \),

\[
 f \succeq g \Rightarrow f \succeq \alpha f + (1 - \alpha)g \succeq g \\
 f \succ g \Rightarrow f \succ \alpha f + (1 - \alpha)g \succ g.
\]
\( p^*(\pi, \tau) \) generates a BPEU representation for the same preference \( \succeq \). Note that the restrictions \( p_\pi(\pi, \tau) \) and \( p_{\pi'}(\pi', \tau') \) need not be equal or related in any other way when the pairs \((\pi, \tau)\) and \((\pi', \tau')\) are distinct.\(^8\)

Transitivity and monotonicity provide additional constraints for the belief function \( p : \Pi \times \Pi \rightarrow \Delta \). In general, the beliefs \( p(\pi, \tau) \) and \( p(\pi', \tau') \) need not equal each other, even if restricted to \( \pi \) or \( \tau \). Monotonicity guarantees the symmetry of the belief function \( p(\pi, \tau) = p(\tau, \pi) \) for all partitions \( \pi \) and \( \tau \).

In the transitive case, the equality \( p(\pi, \tau) = p(\pi, \pi) \) asserts that the beliefs \( p(\pi, \tau) \) can be framed by \( \pi \), but must be invariant of the second variable partition \( \tau \). In this case, BPEU representation (10) becomes a utility representation

\[
U(f) = u(f(q(\pi f)))
\]

where \( q : \Pi \rightarrow \Delta \) is a univariate belief function such that \( q(\pi) = p(\pi, \pi) \) for all \( \pi \in \Pi \). The combination of monotonicity and transitivity turns (10) into the standard expected utility model.

### 3.2 Bi-partitional Maxmin Expected Utility (BPMEU)

Suppose that the preference \( \succeq \) can exhibit framing effects together with ambiguity aversion. Then regularity and RM are still plausible. These conditions imply the general model (8) with some belief function \( b : \mathcal{H} \times \mathcal{H} \rightarrow \Delta \). Assume that the DM’s ambiguity attitudes can be framed by the generated partitions \( \pi f \) and \( \pi g \), but must be invariant of all other aspects of the feasible prospects \( f \) and \( g \). Then B1 should still hold.

However, B2 is problematic for ambiguity averse preferences because varying the stakes in the prospect \( f \) can affect the belief \( b(f, g) \) even if the partitions \( \pi f \) and \( \pi g \) are unchanged. For example, the binary bets \( xEy \) and \( yEx \) for some vaguely understood event \( E \subset S \) and lotteries \( x \succ y \) generate the same partition \( \pi = \{E, S \setminus E\} \). However, ambiguity aversion suggests that the belief \( b(xEy, g) \) should assign a lower probability to \( E \) than \( b(yEx, g) \).

To accommodate ambiguity aversion, relax B2 to

\[\text{B3: } \text{if } f' \in I f, \text{ then } f(b(f, g)) \preceq f(b(f', g)).\]

This property assumes roughly that if \( \pi f = \pi f' \), then any distinction between beliefs \( b(f, g) \) and \( b(f', g) \) should be due to ambiguity aversion rather than framing effects. Similarly to the multiple priors model, assume that the belief \( b(f, g) \) must

---

\(^8\)The total numbers of partitions of \( n \)-element sets are given by the Bell numbers \( B_n \), so that \( B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203 \), etc. Given any state \( s \in S \), there are \( B_{S-1} \) partitions such that \( \{s\} \in \pi \) and hence, \( B_S B_{S-1} \) distinct pairs \( (\pi, \tau) \) for which \( \{s\} \in \pi \). Thus the probability \( p(\pi, \tau)(s) \) can take up to \( B_S B_{S-1} \) distinct values for various \( \pi \) and \( \tau \). The number \( B_S B_{S-1} = O(S^2 e^S) \) grows exponentially with the size of the state space \( S \).
be less favorable for \( f \) than any alternative scenario \( b(f', g) \) that can be used when the partitions \( \pi f' = \pi f \) and \( \pi g \) are unchanged.

I show below (Section 4.5) that B1 and B3 imply Axioms 1–4 and Axiom 6 (Partition-Preserving Uncertainty Aversion (PPUA)). For all \( \alpha \in [0, 1] \) and \( f, g, h \in \mathcal{H} \) such that \( g \in \mathcal{I} f \) and \( \alpha f + (1-\alpha)g \in \mathcal{I} f \),

\[
f \succeq h \quad \text{and} \quad g \succeq h \implies \alpha f + (1-\alpha)g \succeq h.
\]

Obviously, PPUA relaxes PPB. For transitive preferences, this condition follows also from Gilboa–Schmeidler’s Uncertainty Aversion.

Let \( C \) be the set of all non-empty, closed, and convex subsets \( M \subset \Delta \). For any \( \pi \in \Pi \) and \( M \in C \), let

\[
M_\pi = \{ q_\pi \in \Delta_\pi : q \in M \}
\]

be the set of all restrictions \( q_\pi \) of probability measures \( q \in M \) to the partition \( \pi \).

**Theorem 3.** \( \succeq \) satisfies Axioms 1-4 and PPUA iff it is represented by

\[
f \succeq g \iff \min_{q \in M(\pi f, \pi g)} u(f(q)) \geq \min_{q \in M(\pi g, \pi f)} u(g(q)) \tag{11}
\]

where \( u \in \mathcal{U} \) and \( M(\pi, \tau) \in C \) for all \( \pi, \tau \in \Pi \). Moreover,

(i) \( u \) is unique up to a plt, and the set \( M_\pi(\pi, \tau) \) is unique for all \( \pi, \tau \in \Pi \),

(ii) \( \succeq \) is monotonic iff \( \succeq \) is represented by (11) where \( M(\pi, \tau) = M(\tau, \pi) \) for all \( \pi, \tau \in \Pi \),

(iii) \( \succeq \) is transitive iff \( \succeq \) is represented by (11) where \( M(\pi, \tau) = M(\pi', \tau') \) for all \( \pi, \tau, \pi', \tau' \in \Pi \),

(iv) \( \succeq \) is monotonic and transitive iff \( \succeq \) is represented by (11) where \( M(\pi, \tau) = M(\pi', \tau) \) for all \( \pi, \tau, \pi', \tau' \in \Pi \),

This theorem is my main result: all other representations are derived as special cases of (11). The proof of Theorem 3 is sketched in Section 4.4; its details are relegated to the appendix.

The DM as portrayed by (11) compares any pair of acts \( f, g \in \mathcal{H} \) via the least favorable probabilistic scenarios selected from the sets \( M(\pi f, \pi g) \) and \( M(\pi g, \pi f) \) respectively. These sets can be framed by the pair of the partitions \( \pi = \pi f \) and \( \tau = \pi g \) generated by the feasible prospects \( f \) and \( g \). Accordingly, representation (11) is called bi-partitional maxmin expected utility (BPMEU).

The uniqueness claim in Theorem 3 asserts that \( \succeq \) has two distinct representations (11) with components \((u, M)\) and \((u^*, M^*)\) where \( u, u^* \in \mathcal{U} \) and \( M, M^* : \Pi \times \Pi \to C \) if and only if \( u^* \) is a plt of \( u \), and \( M^*_\pi(\pi, \tau) = M_\pi(\pi, \tau) \) for all \( \pi, \tau \in \Pi \). Therefore, the preference \( \succeq \) identifies each restriction \( M_\pi(\pi, \tau) \) uniquely.
on the corresponding partition \( \pi \), but does not put any further constraints on the set \( M(\pi, \tau) \subset \Delta \) besides the technical closedness and convexity. Note that the restrictions \( M_\pi(\pi, \tau) \) and \( M_\pi(\pi', \tau') \) can be arbitrary distinct convex and closed subsets of \( \Delta_\pi \) when the pairs \((\pi, \tau)\) and \((\pi', \tau')\) are distinct.

Transitivity and monotonicity impose additional constraints on the sets \( M(\pi, \tau) \). Monotonicity implies that partitions \( \pi \) and \( \tau \) have symmetric framing effects on the sets of beliefs \( M(\pi, \tau) = M(\tau, \pi) \). In the transitive case, \( \succeq \) has a utility representation

\[
U(f) = \min_{q \in M^*(\pi f)} u(f(q))
\]

for some univariate function \( M^* : \Pi \to \mathbb{C} \). The combination of monotonicity and transitivity turns (11) into the standard MEU model.

### 3.3 Cross-Partitional Representations (CPEU and CPMEU)

When comparing prospects \( f \) and \( g \), the DM may consider their joint descriptions written in terms of non-empty events that have the form

\[
\{ s \in S : f(s) = x \text{ and } g(s) = y \}
\]

for some \( x \) and \( y \). These events comprise the cross-partition \( \pi f \lor \pi g \).

Suppose that for all \( f, g, h \in \mathcal{H} \),

\[
B_1^* : \text{ if } \pi f \lor \pi h = \pi f \lor \pi g, \text{ then } b(f, g) = b(f, h).
\]

Clearly, \( B_1^* \) strengthens \( B_1 \). It assumes that the alternative \( g \) can frame the evaluation of \( f \) only via the events in the cross-partition \( \pi f \lor \pi g \).

\( B_1^* \) implies another weaker version of transitivity

**Axiom 7 (Mixture Transitivity (MT)).** For all \( f, g \in \mathcal{H} \), \( \alpha \in (0, \frac{1}{2}) \), and \( x \in X \),

\[
f \succeq \alpha f + \alpha g + (1 - 2\alpha)x \succeq g \quad \Rightarrow \quad f \succeq g.
\]

Indeed, let \( h = \alpha f + \alpha g + (1 - 2\alpha)x \). Then the equality \( \pi f \lor \pi g = \pi g \lor \pi h = \pi f \lor \pi h \) holds (see Lemma 8 below). By (8) and \( B_1^* \), \( f \succeq h \succeq g \) implies

\[
f(b(f, g)) = f(b(h, f)) \succeq h(b(h, f)) = h(b(h, g)) \succeq g(b(g, h)) = g(b(g, f))
\]

and hence, \( f \succeq g \).

**Theorem 4.** \( \succeq \) satisfies Axioms 1-4, PPUA, and MT iff \( \succeq \) is represented by (11) where

\[
M(\pi, \tau) = M(\pi, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi.
\]

Moreover,
(i) \( \succeq \) is monotonic iff \( \succeq \) has a BPMEU representation (11) where
\[
M(\pi, \tau) = M(\pi \lor \tau, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi, \tag{13}
\]

(ii) \( \succeq \) satisfies PPB iff \( \succeq \) has a BPEU representation (10) where
\[
p(\pi, \tau) = p(\pi, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi.
\]

(iii) \( \succeq \) is monotonic and satisfies PPB iff \( \succeq \) has a BPEU representation (10) where
\[
p(\pi, \tau) = p(\pi \lor \tau, \pi \lor \tau) \quad \text{for all } \pi, \tau \in \Pi.
\]

Theorem 4 identifies special cases of BPMEU and BPEU representations where the beliefs \( p(\pi, \tau) \) and sets of beliefs \( M(\pi, \tau) \) exhibit equal framing effects for the partitions \( \tau \) and \( \pi \lor \tau \). The structures (12) and (13) are called cross-partitional maxmin expected utility (CPMEU) and cross-partitional expected utility (CPEU) respectively.

In the monotonic case, CPEU and CPMEU representations become
\[
f \succeq g \iff u(f(p^*(\pif \lor \pi g))) \geq u(g(p^*(\pif \lor \pi g))),
\]
\[
f \succeq g \iff \min_{q \in \mathcal{M}^*(\pif \lor \pi g)} u(f(q)) \geq \min_{q \in \mathcal{M}^*(\pif \lor \pi g)} u(g(q))
\]
for some univariate functions \( p^* : \Pi \to \Delta \) and \( M^* : \Pi \to \mathcal{C} \) respectively. The monotonic CPEU representation is analogous to Ahn and Ergin’s partition-dependent expected utility (PDEU), which I further discuss in Section 4.3.

Moreover, the cross-partitional structure can be naturally applied to accommodate framing in choices among more than two alternatives. Let \( \mathcal{M} \) be the set of all menus—non-empty finite subsets \( F \subset \mathcal{H} \). A choice function \( c : \mathcal{M} \to \mathcal{M} \) specifies a non-empty set \( c(F) \subset \mathcal{A} \) of all acts that the DM may be willing to choose in a given menu \( A \in \mathcal{M} \). Suppose that a choice function \( c(\cdot) \) is given as a primitive.

For each \( F \in \mathcal{M} \), let \( \pi F = \pi f_1 \lor \pi f_2 \lor \cdots \lor \pi f_n \) be the cross-partition generated by all feasible acts in a menu \( F = \{f_1, \ldots, f_n\} \). Then CPEU and CPMEU representations can be used to describe the function \( c(A) \) for all menus \( A \in \mathcal{M} \) as
\[
c(F) = \arg \max_{f \in F} u(f(p(\pi f, \pi F))),
\]
\[
c(F) = \arg \max_{f \in F} \min_{q \in \mathcal{M}^*(\pi f, \pi F)} u(f(q))
\]
respectively.
4 Discussion

Next, I show how the bi-partitional and cross-partitional framing representations can describe source dependence, comparative ignorance, and preference reversals. I also clarify the formal connection of CPEU to Ahn and Ergin’s partition-dependent expected utility (PDEU) model and Tversky’s support theory. Finally, I outline the proofs.

4.1 Unframable Beliefs and Source-Dependence

Here I address the following natural question: under what conditions all the revealed subjective beliefs or sets of beliefs in the BPEU or BPMEU models should agree on a given partition $\pi^* \in \Pi$?

Say that a partition $\pi^* \in \Pi$ is unframable if for all $f \in H_{\pi^*}$, $g, h \in H$, and $x \in X$,

\begin{align*}
  g \geq f \succeq x \geq h & \Rightarrow g \geq h \\
  g \geq x \succeq f \geq h & \Rightarrow g \succeq h.
\end{align*}

This definition assumes roughly that if $f$ is measurable with respect to an unframable partition $\pi^*$, then

(a) the evaluation $V^* = u(f(b(f, \cdot)))$ of $f$ should be unaffected by framing,

(b) the evaluations $V_g = u(g(b(g, h)))$ of any act $g \geq f$ cannot be framed to be below $V^*$, regardless of the alternative $h$,

(c) the evaluation $V_h = u(h(b(h, g)))$ of any act $h \leq f$ cannot be framed to be above $V^*$.

Clearly, all of these assumptions hold in the BPMEU model where the sets $M(\pi, \tau)$ have the same restrictions $M_{\pi^*}(\pi, \tau)$ for all $\pi, \tau \in \Pi$.

It follows from (a)-(c) that $g \geq f \succeq x \geq h$ implies $V_g \geq V^* \geq u(x) \geq V_h$, and $g \geq x \succeq f \geq h$ implies $V_g \geq u(x) \geq V^* \geq V_h$. In each case, $V_g \geq V_h$ implies $g \succeq h$.

**Theorem 5.** Suppose that $\succeq$ satisfies Axioms 1–4 and PPUA. Then $\pi^*$ is unframable iff $\succeq$ has a BPMEU representation (11) where the function $M : \Pi \times \Pi \rightarrow \mathcal{C}$ is such that

$$M_{\pi^*}(\pi, \tau) = M_{\pi^*}(\pi', \tau')$$

for all $\pi, \tau, \pi', \tau' \in \Pi$.

Moreover, if (11) satisfies PPB, then $\pi^*$ is unframable iff $\succeq$ has a BPEU representation (10), where the belief function $p : \Pi \times \Pi \rightarrow \mathcal{C}$ is such that

$$p_{\pi^*}(\pi, \tau) = p_{\pi^*}(\pi^*, \pi^*)$$

for all $\pi, \tau \in \Pi$.

This result shows that definition (14) allows to combine the standard multiple priors and expected utility models on the domain of all $\pi^*$ measurable acts with framing in more general choices.
4.2 Comparative Ignorance and Preference Reversals

Comparative ignorance is another natural application of the BPMEU model.

Given any two partitions \( \pi, \tau \in \Pi \), say that \( \pi \)

- decreases confidence in \( \tau \) if for all \( f \in \mathcal{H}_\pi \), \( g \in \mathcal{H}_\tau \), and \( x \in X \),
  \[
  f \succeq x \succeq g \implies f \succeq g,
  \]

- increases confidence in \( \tau \) if for all \( f \in \mathcal{H}_\pi \), \( g \in \mathcal{H}_\tau \), and \( x \in X \),
  \[
  g \succeq x \succeq f \implies g \succeq f.
  \]

Note that the definition is not symmetric. If \( \pi \) increases confidence \( \tau \), then \( \tau \) can increase, or decrease, or have more complicated effects on the confidence in \( \pi \).

Say that the partitions \( \pi \) and \( \tau \) reveal comparative ignorance if for all \( f \in \mathcal{H}_\pi \), \( g \in \mathcal{H}_\tau \), and \( x \in X \),
\[
f \succeq x \succeq g \implies f \succeq g.
\]

In the BPMEU model, each act \( f \in \mathcal{H} \) has some certainty equivalent \( c(f) \in X \). Note that the ranking \( f \succeq x \succeq g \) holds if and only if \( c(f) \succeq c(g) \). Thus the above definition of comparative ignorance prohibits preference reversals \( c(f) \succeq c(g) \) and \( g \succ f \) when \( f \) and \( g \) generate partitions \( \pi \) and \( \tau \) respectively.

Let \( \pi_0 = \{\emptyset, S\} \) be the coarsest partitions of \( S \).

**Theorem 6.** Suppose that \( \succeq \) has a BPMEU representation (11). Then for any \( \pi, \tau \in \Pi \),

(i) \( \pi \) decreases confidence in \( \tau \) iff \( M_\tau(\tau, \pi_0) \subset M_\tau(\tau, \pi) \),

(ii) \( \pi \) increases confidence in \( \tau \) iff \( M_\tau(\tau, \pi_0) \supset M_\tau(\tau, \pi) \),

(iii) \( \pi \) and \( \tau \) reveal comparative ignorance iff \( M_\pi(\pi, \pi_0) \subset M_\pi(\pi, \tau) \) and \( M_\tau(\tau, \pi_0) \subset M_\tau(\tau, \pi) \).

Thus the above definition of comparative ignorance is equivalent to the requirement that \( \pi \) decreases confidence in \( \tau \), but \( \tau \) increases confidence in \( \pi \).

4.3 CPEU and Support Theory

Recall that the monotonic CPEU model has the form
\[
f \succeq g \iff u(f(p^*(\pi f \lor \pi g))) \geq u(g(p^*(\pi f \lor \pi g)))
\]

where \( p^* : \Pi \to \Delta \).

Representation (15) is tightly related to the partition-dependent expected utility model (PDEU) proposed by Ahn and Ergin [1]. In their setting, preferences
\[
\succeq^* \text{ are defined over pairs } (f, \pi) \in H \times \Pi \text{ such that } \pi \text{ refines } \pi f. \text{ This primitive is used to derive another binary relation } \succeq \text{ on } H \text{ as }
\]
\[
f \succeq g \iff (f, \pi f \lor \pi g) \succeq^* (g, \pi f \lor \pi g).
\]
In my setting, \(\succeq\) is observable, and \(\succeq^*\) can be derived via representation (15) as
\[
(f, \pi) \succeq^* (g, \tau) \iff u(f(p^*(\pi \lor \tau))) \geq u(g(p^*(\pi \lor \tau))). \tag{16}
\]
In addition to (16), Ahn and Ergin characterize additional structure for the belief function \(p^* : \Pi \rightarrow \Delta\): for any partition \(\pi = \{E_1, \ldots, E_n\}\) and event \(E_i \in \pi\),
\[
p(E_i, \pi) = \frac{\nu(E_i)}{\sum_{j=1}^n \nu(E_j)} \tag{17}
\]
for some support function \(\nu : 2^S \rightarrow \mathbb{R}^+\) such that \(\sum_{E_j \in \pi} \nu(E_j) > 0\).

The required axioms are

**Axiom 8** (The Sure-Thing Principle (STP)). For all acts \(f, g, h, h' \in H\) and events \(E \subset S\),
\[
f E h \succeq g E h \implies f E h' \succeq g E h'.
\]

**Axiom 9** (Binary Bet Acyclicity (BBA)). For any cycle of events \(E_1, E_2, \ldots, E_n, E_1\) such that \(E_1 \cap E_2 = E_2 \cap E_3 = \cdots = E_{n-1} \cap E_n = E_n \cap E_1 = \emptyset\), and for any lotteries \(x_1, \ldots, x_n, y \in X\),
\[
x_1 E_1 y \succ x_2 E_2 y \succ \cdots \succ x_n E_n y \implies x_1 E_1 y \succeq x_n E_n y.
\]

The same axioms can be applied to the weak preference \(\succeq\) in my model.

**Corollary 7.** A monotonic preference \(\succeq\) satisfies Axioms 2-5, MT, STP, and BBA iff it is represented by (15), where \(u \in U\) and the belief function \(p^* : \Pi \rightarrow \Delta\) has structure (17) for some \(\nu : 2^S \rightarrow \mathbb{R}^+\) such that \(\sum_{E_j \in \pi} \nu(E_j) > 0\) for all \(\pi \in \Pi\).

**Proof.** Suppose that \(\succeq\) satisfies the required axioms. By Theorem 4, it has representation (15). Define \(\succeq^*\) via (16). Then it satisfies Ahn and Ergin’s model. By their Theorem 3, the belief function \(p^*\) has the required structure. \(\square\)

### 4.4 Sketch of Proofs: Axioms \(\Rightarrow\) Representations

The central part of the proofs is the construction of the sets \(M(\pi, \tau)\) in the BP-MEU representation (11) in Theorem 3. Suppose that \(\succeq\) satisfies Axioms 1-4 and PPUA.

Transitivity, Archimedean, and Independence on the domain of lotteries \(X\) follow from PPT, CA, and CI respectively. Thus \(\succeq\) is represented on \(X\) by an
expected utility function \( u : X \to \mathbb{R} \) (see Theorem 5.4 in Kreps [10]). If \( u \) is constant, then by RM, \( \succcurlyeq \) is empty. Hence \( u \in \mathcal{U} \) is not constant.

Wlog, assume that \([-1, 1] \subset u(X)\). Take \( c_0 \in X \) such that \( u(c_0) = 0 \).

For any \( f, g \in \mathcal{H} \), write \( f \gg g \) if \( f(s) > g(s') \) for all \( s, s' \in S \). If \( f \gg g \), then by RM \( f \succ g \) must hold.

For any act \( f \in \mathcal{H} \) and partitions \( \pi, \tau \in \Pi \), let

\[
V(f, \pi, \tau) = \sup_{(\alpha, z) \in Z(f, \pi, \tau)} \frac{u(z)}{\alpha}
\]  

where \( Z(f, \pi, \tau) \) is the set of all pairs \( (\alpha, z) \in (0, 1] \times X \) such that

\[
\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f_\tau \gg z \quad \text{for some } f_\pi \in \mathcal{H}_\pi \text{ and } f_\tau \in \mathcal{H}_\tau.
\]

Formula (18) uses the approximations \( f_\pi \) and \( f_\tau \) to evaluate the act \( f \) under the framing partitions \( \pi \) and \( \tau \).

I claim that for any \( \pi, \tau \in \Pi \), there is a set \( M(\pi, \tau) \in \mathcal{C} \) such that

\[
V(f, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(f(q)) \quad \text{for all } f \in \mathcal{H}.
\]  

In this way, the function \( V : \mathcal{H} \times \Pi \times \Pi \to \mathbb{R} \) identifies all of the required sets \( M(\pi, \tau) \). The maxmin representation (19) is derived via a sequence of Lemmas 9-11 in the appendix.

Next I show (Lemma 12) that for all \( f, g \in \mathcal{H} \),

\[
f \succeq g \iff V(f, \pi f, \pi g) \geq V(g, \pi g, \pi f).
\]  

The combination of (19) and (20) implies BPMEU representation (11).

The last step is to derive various special cases of BPMEU representation where

- \( \succeq \) is monotonic, and the symmetry \( M(\pi, \tau) = M(\tau, \pi) \) holds,
- \( \succeq \) is transitive, and the function \( M(\pi, \tau) = M(\pi, \pi) \) is univariate,
- \( \succeq \) satisfies PPB, and the sets \( M(\pi, \tau) = \{p(\pi, \tau)\} \) are singletons,
- \( \succeq \) satisfies MT, and the cross-partitional structure \( M(\pi, \tau) = M(\pi, \pi \lor \tau) \) holds.

4.5 Proofs: Representations ⇒ Axioms

It is also instructive to derive the axioms from representations in Theorems 2–4. Start with the BPMEU representation (11) in Theorem 3.

Assume that \( \succeq \) is represented by (11). For each \( f \in \mathcal{H} \) and \( \tau \in \Pi \), fix a belief

\[
b^*(f, \tau) \in \arg \min_{q \in M(\pi f, \tau)} u(f(q)).
\]
For any \( g \in \mathcal{H} \), assign \( b(f,g) = b^*(f,\pi g) \). By construction, the beliefs \( b(f,g) \) satisfy B1, B3, and representation (8).

**CI and CA**: Take any \( f, g \in \mathcal{H} \) and \( x \in X \). For all \( \alpha \in (0,1) \), let \( f_\alpha = \alpha f + (1-\alpha)x \) and \( g_\alpha = \alpha g + (1-\alpha)x \). Let \( p = b(f,g), q = b(g,f) \), \( p_\alpha = b(f_\alpha, g_\alpha) \), and \( q_\alpha = b(g_\alpha, f_\alpha) \). By B1, \( p_\alpha = b(f_\alpha, g) \) and \( q_\alpha = b(g_\alpha, f) \). By B3, \( f(p) \preceq f(p_\alpha) \) and \( f_\alpha(p_\alpha) \preceq f_\alpha(p) \). Thus \( f(p) \sim f(p_\alpha) \). Similarly, \( g(q) \sim g(q_\alpha) \). By (8),

\[
\begin{align*}
f \succeq g &\iff f(p) \succeq g(q) \iff f_\alpha(p_\alpha) \succeq g_\alpha(q_\alpha) \iff f_\alpha \succeq g_\alpha,
\end{align*}
\]

and CI holds. Moreover, if \( f(p) \succ g(q) \), then \( \alpha f_\alpha + (1-\alpha)x \succ (\prec) g_\alpha(q) \) and \( f(p) \succ \beta g_\alpha(q) + (1-\beta)x \) or \( \beta \) for some \( \alpha, \beta \in (0,1) \). Thus CA holds.

**PPT**: Take any \( f, g \in \mathcal{H}, f' \in \mathcal{I} f \) and \( g' \in \mathcal{I} g \). By B1, \( p = b(f,g) = b(f,g') \), \( p' = b(f',g) = b(f',g') \), \( q = b(g,f) = b(g,f') \) and \( q' = b(g',f) = b(g',f') \) for some \( p, p', q, q' \in \Delta \). By (8) and B3,

\[
\begin{align*}
f \succeq g' &\iff f(p) \succeq g'(q') \iff f'(p') \succeq g'(q') \iff f \succeq g
\end{align*}
\]

**PPUA**: Take any \( f, h, g \in \mathcal{H}, g \in \mathcal{I} f \), and \( \alpha \in [0,1] \) such that \( f_\alpha = \alpha f + (1-\alpha)g \in \mathcal{I} f \). Suppose that \( f \succeq h \) and \( g \succeq h \). Let \( p_1 = b(h,f) \), \( p_\alpha = b(f_\alpha, h) \), \( p_0 = b(g,h) \). By B1, \( b(h,f) = b(h,g) = b(h, f_\alpha) = q \) for some \( q \in \Delta \). By B3, \( f(p_\alpha) \succeq f(p_1) \) and \( g(p_\alpha) \succeq g(p_0) \). Then

\[
f_\alpha(p_\alpha) = \alpha f_\alpha(p_\alpha) + (1-\alpha)g(p_\alpha) \succeq \alpha f_\alpha(p_1) + (1-\alpha)g(p_0) \succeq \alpha h(q) + (1-\alpha)h(q) = h(q).
\]

Thus \( f_\alpha \succeq h \).

**PPB**: In the BPEU case, \( b(f,g) = p(\pi f, \pi g) \) and hence, B2 holds. The equality \( p_1 = p_\alpha = p_0 \) is implied by B2. By (8), \( f(p_1) \preceq h(q) \) and \( g(p_0) \preceq h(q) \) imply \( f_\alpha(p_\alpha) \preceq h(q) \), that is, \( f_\alpha \preceq h \). Thus PPB holds.

**MT**: In the CPMEU case, take the belief

\[
b^*(f, \tau) \in \arg \min_{q \in \mathcal{M}(\pi f, \pi \tau \vee \pi f)} u(f(q))
\]

and for any \( g \in \mathcal{H} \), assign \( b(f,g) = b^*(f,\pi g) \). Then the beliefs \( b(f,g) \) satisfy B1* and hence, MT.

CPEU is the overlap of BPEU and CPMEU and hence, satisfies all of the above axioms.

**A. APPENDIX: PROOFS**

Show two properties of cross partitions.

**Lemma 8.** For all \( f, g \in \mathcal{H} \) and \( \alpha \in (0,1) \),

\[
\pi f \vee \pi (\alpha f + (1-\alpha)g) = \pi f \vee \pi g.
\]  

(21)

Moreover, the set \( I = \{ \gamma \in (0,1) : \pi (\gamma f + (1-\gamma)g) \neq \pi f \vee \pi g \} \) is finite.
Proof. Fix any \( f, g \in \mathcal{H} \) and a mixture \( h = \alpha f + (1 - \alpha)g \) for \( \alpha \in (0, 1) \). Show that

\[
\bullet \; \pi f \vee \pi g = \pi f \vee \pi h,
\]

\[
\bullet \; \pi f \vee \pi g = \pi h \text{ whenever } \alpha \not\in I \text{ for some finite } I \subset (0, 1).
\]

Take any \( E \in \pi f \vee \pi g \). Then \( E = f^{-1}(x) \cap g^{-1}(y) \neq \emptyset \) for some \( x, y \in X \). Let \( E' = f^{-1}(x) \cap h^{-1}(\alpha x + (1 - \alpha)y) \). Then \( E' \in \pi f \vee \pi h \) and \( E' = E \).

Take any \( E' \in \pi f \vee \pi h \). Then \( E' = f^{-1}(x) \cap h^{-1}(z) \) for some \( x, z \in X \). Then \( z = \alpha x + (1 - \alpha)g(s) \) for some \( s \in S \). Let \( y = g(s) \). Then \( E' = f^{-1}(x) \cap g^{-1}(y) \in \pi f \vee \pi g \). Thus \( \pi f \vee \pi g = \pi f \vee \pi h \).

Next, let \( f(S) = \{x_1, \ldots, x_n\} \) and \( g(S) = \{y_1, \ldots, y_r\} \) be the finite ranges of the acts \( f \) and \( g \) with \( x_i \neq x_j \) and \( y_i \neq y_j \) for all \( i \neq j \). Let

\[
Y_\alpha = \{\alpha x_i + (1 - \alpha)y_j : i = 1, \ldots, n \text{ and } j = 1, \ldots, r\}
\]

be the set of all distinct mixtures \( \alpha x_i + (1 - \alpha)y_j \). Note that if \( Y_\alpha \) has \( nr \) elements, then \( \pi f \vee \pi g = \pi h \).

Fix any indices \( i, k \in \{1, \ldots, n\} \) and \( j, m \in \{1, \ldots, r\} \). Suppose that the pairs \((i, j)\) and \((k, m)\) are distinct. Then \( x^* = x_i - x_k \) and \( y^* = y_m - y_j \) are two signed measures on \( D \) that are not both zero. Then the equality \( \alpha x^* = (1 - \alpha)y^* \) can hold for at most one value \( \alpha = \alpha_{ijkm} \in (0, 1) \). Let \( I \) be the finite set of all such values \( \alpha_{ijkm} \). Thus the set \( Y_\alpha \) consists fewer than \( nr \) elements only if \( \alpha \in I \). \( \square \)

Note that for all \( f, g \in \mathcal{H} \), \( \alpha \in (0, 0.5) \) and \( x \in X \),

\[
\alpha f + \alpha g + (1 - 2\alpha)x = 2\alpha \left( \frac{f + g}{2} \right) + (1 - 2\alpha)x.
\]

Thus Lemma 8 implies \( \pi f \vee \pi g = \pi f \vee \pi h = \pi g \vee \pi h \) where \( h = \alpha f + \alpha g + (1 - 2\alpha)x \).

**Proof of Theorem 3**

Suppose that \( \succ \) is not empty, \( \succeq \) is complete and satisfies Axioms 1–5.

PPT, CA, and CI imply that \( \succeq \) satisfies Transitivity, Archimedean, and Independence on the domain of lotteries \( X \). Thus \( \succeq \) is represented on \( X \) by an expected utility function \( u : X \to \mathbb{R} \) (see Theorem 5.4 in Kreps [10]). If \( u \) is constant, then by RM, \( \succ \) is empty. Hence \( u \in \mathcal{U} \) is not constant.

Note that for any \( f \in \mathcal{H}, z \in X, \) and \( \tau \in \Pi \),

\[
f \gg z \implies f \gg h_\tau \gg z \text{ for some } h_\tau \in \mathcal{H}_\tau.
\]

(22)

Indeed, if \( f \gg z \), then there is \( y \in X \) such that \( f \gg y \gg z \) because \( f \) has finite range and its worst outcome is strictly better than \( z \). Take any partition \( \tau = \{E_1, \ldots, E_k\} \). Define an act \( h_\tau(s) = \frac{1}{1 + \tau} y + \frac{\tau}{1 + \tau} z \) for all \( i = 1, \ldots, k \) and \( s \in E_i \). Then \( \pi h_\tau = \tau \) and \( f \gg y \gg h_\tau \gg z \). Thus \( f \gg h_\tau \gg z \).
Wlog, assume that \([-1, 1] \subset u(X)\). Let \(c_0 \in X\) such that \(u(c_0) = 0\).

For any act \(f \in \mathcal{H}\) and partitions \(\pi, \tau \in \Pi\), let

\[
V(f, \pi, \tau) = \sup_{(\alpha, z) \in Z(f, \pi, \tau)} \frac{u(z)}{\alpha}
\]  

(23)

where \(Z(f, \pi, \tau)\) is the set of all pairs \((\alpha, z) \in (0, 1] \times X\) such that

\[
\alpha f + (1 - \alpha) c_0 \geq f_\pi \succ f_\tau \succ z \quad \text{for some } f_\pi \in \mathcal{H}_\pi \text{ and } f_\tau \in \mathcal{H}_\tau.
\]  

(24)

The function \(V : \mathcal{H} \times \Pi \times \Pi \to \mathbb{R}\) identifies the required BPMEU representation (11) via a sequence of lemmas.

**Lemma 9.** The function \(V\) is well-defined and monotonic: for all \(f, g \in \mathcal{H}\),

\[
f \succeq g \quad \Rightarrow \quad V(f, \pi, \tau) \geq V(g, \pi, \tau).
\]  

(25)

**Proof.** Fix any \(\pi, \tau \in \Pi\). Show that for all \(x \in X\),

\[
V(x, \pi, \tau) = u(x).
\]  

(26)

Take any \(x \in X\). As \(u\) is not constant, then \(0.5x + 0.5c_0 \succ y \succ z\) for some \(y, z \in X\). Take \(f_\pi \in \mathcal{H}_\pi\) and \(f_\tau \in \mathcal{H}_\tau\) such that

\[
0.5x + 0.5c_0 \gg f_\pi \gg y \gg f_\tau \gg z.
\]

By RM, \(f_\pi \succ f_\tau\). Thus \((0.5, z) \in Z(x, \pi, \tau)\). By (23), \(V(x, \pi, \tau) \geq 2u(z)\). As \(u(z)\) can be taken arbitrarily close to \(u(0.5x + 0.5c_0)\), then

\[
V(x, \pi, \tau) \geq 2u(0.5x + 0.5c_0) = u(x).
\]

Conversely, take any \((\alpha, z) \in Z(x, \pi, \tau)\), and \(f_\pi \in \mathcal{H}_\pi\), \(f_\tau \in \mathcal{H}_\tau\) such that

\[
\alpha x + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z.
\]

If \(z \succ \alpha x + (1 - \alpha)c_0\), then \(f_\tau \gg z \succ \alpha x + (1 - \alpha)c \geq f_\pi\) and hence, \(f_\tau \gg f_\pi\). By RM, \(f_\tau \succ f_\pi\), which contradicts \(f_\pi \succ f_\tau\). By contradiction, \(\alpha x + (1 - \alpha)c_0 \geq z\).

Thus \(\alpha u(x) \geq u(z)\) and \(u(x) \geq V(x, \pi, \tau)\).

For any \(f \in \mathcal{H}\), fix \(x, y \in X\) such that \(x \geq f \geq y\). By (24),

\[
Z(x, \pi, \tau) \supset Z(f, \pi, \tau) \supset Z(y, \pi, \tau).
\]

Thus \(Z(f, \pi, \tau)\) is not empty. By (23) and (26), \(V(f, \pi, \tau) \leq V(x, \pi, \tau) = u(x)\) is well-defined.

Similarly, if \(f \succeq g\), then \(Z(f, \pi, \tau) \supset Z(g, \pi, \tau)\). Thus, (23) implies (25). \(\square\)
**Lemma 10.** For all $\gamma \in [0, 1]$, $x \in X$, and $f \in \mathcal{H}$,
\[
V(\gamma f + (1 - \gamma)x, \pi, \tau) = \gamma V(f, \pi, \tau) + (1 - \gamma)u(x). \tag{27}
\]

*Proof.* Fix any $\pi, \tau \in \Pi$. If $\gamma = 0$, then (27) follows from (26).

Take any $\gamma \in (0, 1]$, $f \in \mathcal{H}$, and $x \in X$. Let $g = \gamma f + (1 - \gamma)x$, and
\[
\alpha g + (1 - \alpha)c_0 \geq g_\pi \succ g_\tau \gg z
\]
for some $(\alpha, z) \in Z(g, \pi, \tau)$, $g_\pi \in \mathcal{H}_\pi$, and $g_\tau \in \mathcal{H}_\tau$. Take $\beta \in (0, 1)$ and $y \in X$ such that
\[
\beta \alpha (1 - \gamma)u(x) + (1 - \beta)u(y) = 0.
\]
Then
\[
g' \geq \beta g_\pi + (1 - \beta)y \succ \beta g_\tau + (1 - \beta)y \geq \beta z + (1 - \beta)y
\]
where $g' = \beta(\alpha g + (1 - \alpha)c_0) + (1 - \beta)y$ satisfies $\beta \alpha \gamma f + (1 - \beta \alpha \gamma)c_0 \geq g'$. Thus
\[
V(f, \pi, \tau) \geq \frac{u(\beta z + (1 - \beta)y)}{\beta \alpha \gamma} = \frac{\beta u(z) - \beta \alpha (1 - \gamma)u(x)}{\beta \alpha \gamma}
\]
that is, $\gamma V(f, \pi, \tau) + (1 - \gamma)u(x) \geq \frac{u(z)}{\alpha}$. As $(\alpha, z) \in Z(g, \pi, \tau)$ is arbitrary, then
\[
\gamma V(f, \pi, \tau) + (1 - \gamma)u(x) \geq V(g, \pi, \tau).
\]

Conversely, suppose that
\[
\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \gg z
\]
for some $(\alpha, z) \in Z(f, \pi, \tau)$, $f_\pi \in \mathcal{H}_\pi$, and $f_\tau \in \mathcal{H}_\tau$. Then
\[
\alpha g + (1 - \alpha)c_0 \geq g_\pi \succ g_\tau \gg y, \quad \text{where}
\]
\[
g_\pi = \gamma f_\pi + (1 - \gamma)(\alpha x + (1 - \alpha)c_0)
\]
\[
g_\tau = \gamma f_\tau + (1 - \gamma)(\alpha x + (1 - \alpha)c_0)
\]
\[
y = \gamma z + (1 - \gamma)(\alpha x + (1 - \alpha)c_0).
\]
Thus $V(g, \pi, \tau) \geq \frac{u(y)}{\alpha} = \gamma \frac{u(z)}{\alpha} + (1 - \gamma)u(x)$, and
\[
V(g, \pi, \tau) \geq \gamma V(f, \pi, \tau) + (1 - \gamma)u(x)
\]
holds because $(\alpha, z) \in Z(f, \pi, \tau)$ is arbitrary. Thus (27) holds. \qed

**Lemma 11.** For all $f, g \in \mathcal{H}$, and $\gamma \in [0, 1]$,
\[
V(\gamma f + (1 - \gamma)g, \pi, \tau) \geq \min\{V(f, \pi, \tau), V(g, \pi, \tau)\}. \tag{28}
\]
Proof. Fix any partitions $\pi, \tau \in \Pi$ and any acts $f, g \in H$.

First, I claim that the function

$$v(\alpha) = V(\alpha f + (1 - \alpha)g, \pi, \tau)$$

is continuous for all $\alpha \in [0, 1]$. To show this claim, fix $x^*, x_*, y^*, y_* \in X$ such that

$$x^* \geq f \geq x_* \text{ and } y^* \geq g \geq y_*.$$ 

By (25) and (27)

$$\alpha u(x^*) + (1 - \alpha)V(g, \pi, \tau) = V(\alpha x^* + (1 - \alpha)g) \geq v(\alpha) \geq V(\alpha x_* + (1 - \alpha)g) \geq \alpha u(x_*) + (1 - \alpha)V(g, \pi, \tau).$$

Thus, $v$ is continuous at $\alpha = 0$. Fix any $\alpha \in (0, 1)$. Let $h = \alpha f + (1 - \alpha)g$.

The above argument implies that the function $w(\beta) = V(\beta g + (1 - \beta)h, \pi, \tau)$ is continuous at $\beta = 0$. Thus the function $v(\alpha - \varepsilon) = w\left(\frac{\varepsilon}{\alpha}\right)$ is continuous at $\varepsilon = 0$. To show continuity of $v(\alpha + \varepsilon)$ at $\varepsilon = 0$, switch $f$ with $g$, and $\alpha$ with $1 - \alpha$. Use the same switch to show that $v(\alpha)$ is continuous at $\alpha = 1$. Thus, $v(\alpha)$ is continuous at any $\alpha \in [0, 1]$.

Proceed to show (28). Take any $\lambda \in \mathbb{R}$ such that

$$\lambda < \min\{V(f, \pi, \tau, V(g, \pi, \tau))\}. \quad (29)$$

By definition of $V$, there are pairs $(\alpha, z_f) \in Z(f, \pi, \tau)$ and $(\beta, z_g) \in Z(g, \pi, \tau)$, and acts $f_\pi, g_\pi \in H_\pi$ and $f_\tau, g_\tau \in H_\tau$ such that

$$\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \succ z_f \quad \text{and} \quad \alpha \lambda < u(z_f)$$

$$\beta g + (1 - \beta)c_0 \geq g_\pi \succ g_\tau \succ z_g \quad \text{and} \quad \beta \lambda < u(z_g).$$

Wlog $\beta \geq \alpha$. Let $g'_\pi = \frac{\alpha}{\beta}g_\pi + \frac{\beta - \alpha}{\beta}c_0$, $g'_\tau = \frac{\alpha}{\beta}g_\tau + \frac{\beta - \alpha}{\beta}c_0$, and $z'_g = \frac{\alpha}{\beta}z_g + \frac{\beta - \alpha}{\beta}c_0$. Then

$$\alpha g + (1 - \alpha)c_0 \geq g'_\pi \succ g'_\tau \succ z'_g \quad \text{and} \quad \alpha \lambda = \frac{\alpha}{\beta}\beta \gamma < \frac{\alpha}{\beta}u(z_g) = u(z'_g).$$

Let $z \in X$ to be the worse of the two lotteries $z_f$ and $z'_g$. Then

$$\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \succ z$$

$$\alpha g + (1 - \alpha)c_0 \geq g'_\pi \succ g'_\tau \succ z,$$

and $\alpha \lambda < u(z)$. Consider two cases.

(i) $f_\tau \succeq g'_\pi$. By PPT, $f_\pi \succ f_\tau \succeq g'_\pi \succ g'_\tau$ implies that $f_\pi \succ g'_\tau$. Take any $\gamma \in [0, 1]$ such that $\gamma f_\pi + (1 - \gamma)g'_\pi \in \mathcal{I}(f_\pi)$. By PPUA,

$$\gamma f_\pi + (1 - \gamma)g'_\pi \succeq g'_\tau.$$
Then by (22), there are \( h_\pi \in H_\tau \) and \( h_\tau \in H_\tau \) such that
\[
g'_\tau \gg h_\pi \gg h_\tau \gg z.
\]

If \( h_\tau \geq \gamma f_\pi + (1 - \gamma)g'_\pi \), then
\[
h_\tau \geq \gamma f_\pi + (1 - \gamma)g'_\pi \geq g'_\tau \geq h_\pi
\]
implies by PPT that \( h_\tau \geq h_\pi \). Yet by RM, \( h_\pi \succ h_\tau \). Thus
\[
\alpha(\gamma f + (1 - \gamma)g) + (1 - \alpha)c_0 \geq \gamma f_\pi + (1 - \gamma)g'_\pi \gg h_\tau \gg z.
\]

Then \( V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \frac{u(z)}{\alpha} > \lambda \).

(ii) \( g'_\pi \succ f_\tau \). Take any \( \gamma \in [0, 1] \) such that \( \gamma f_\pi + (1 - \gamma)g'_\pi \in \mathcal{I}(f_\pi) \). By PPUA,
\[
\gamma f_\pi + (1 - \gamma)g'_\pi \succeq f_\tau.
\]

Then by (22), there are \( h_\pi \in H_\tau \) and \( h_\tau \in H_\tau \) such that
\[
f_\tau \gg h_\pi \gg h_\tau \gg z.
\]

If \( h_\tau \geq \gamma f_\pi + (1 - \gamma)g'_\pi \), then
\[
h_\tau \geq \gamma f_\pi + (1 - \gamma)g'_\pi \geq f_\tau \geq h_\pi
\]
implies by PPT that \( h_\tau \geq h_\pi \). Yet by RM, \( h_\pi \succ h_\tau \). Thus
\[
\alpha(\gamma f + (1 - \gamma)g) + (1 - \alpha)c_0 \geq \gamma f_\pi + (1 - \gamma)g'_\pi \gg h_\tau \gg z.
\]

Then \( V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \frac{u(z)}{\alpha} > \lambda \).

In each case, the inequality \( V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \lambda \) holds for any \( \gamma \) such that \( \gamma f_\pi + (1 - \gamma)g'_\pi \in \mathcal{I}(f_\pi) \). By Lemma 8, this inclusion holds for all \( \gamma \in [0, 1] \setminus I \) where \( I \) is a finite set. By continuity, the function
\[
v(\gamma) = V(\gamma f + (1 - \gamma)g, \pi f, \pi g) \geq \lambda
\]
for all \( \gamma \in [0, 1] \). As \( \lambda \in \mathbb{R} \) is an arbitrary number that satisfies (29), then (28) holds.

**Lemma 12.** For all \( f, g \in H \),
\[
f \succeq g \iff V(f, \pi f, \pi g) \geq V(g, \pi g, \pi f).
\]
Proof. Take any $f, g \in \mathcal{H}$ such that $f \succeq g$. Let $\pi = \pi f$ and $\tau = \pi g$. Take any $\alpha \in (0, 1], z \in X$, $g_\pi \in \mathcal{H}_\pi$, and $g_\tau \in \mathcal{H}_\tau$ such that

$$\alpha g + (1 - \alpha)c_0 \geq g_\tau \succ g_\pi \gg z.$$ 

Let $f_\pi = \alpha f + (1 - \alpha)c_0$. By (22), there are $h_\pi \in \mathcal{H}_\pi$ and $h_\tau \in \mathcal{H}_\tau$ such that $g_\pi \gg h_\pi \gg h_\tau \gg z$. By CI,

$$f_\pi \succeq \alpha g + (1 - \alpha)c_0 \geq g_\tau \succ g_\pi \gg h_\pi \gg h_\tau \gg z.$$ 

By PPT, $\alpha g + (1 - \alpha)c_0 \succeq h_\pi$ and $h_\tau \succeq f_\pi$, then by PPT,

$$h_\tau \succeq f_\pi \succeq \alpha g + (1 - \alpha)c_0 \succeq h_\pi$$

implies $h_\tau \succeq h_\pi$, which contradicts $h_\pi \succ h_\tau$. Thus

$$\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ h_\tau \gg z.$$ 

As $(\alpha, z) \in Z(g, \tau, \pi)$ is arbitrary, then $V(f, \pi, \tau) \geq V(g, \tau, \pi)$.

Suppose that $f \succ g$. Take $x, y \in X$ such that $x \succ y$ and

$$u(x) \geq V(f, \pi, \tau) \geq V(g, \tau, \pi) \geq u(y).$$

By CA, $f \succ \alpha g + (1 - \alpha)x$ and $\beta f + (1 - \beta)y \succ \alpha g + (1 - \alpha)x$ for some $\alpha, \beta \in (0, 1)$. Thus

$$\beta V(f, \pi, \tau) + (1 - \beta)u(y) = V(\beta f + (1 - \beta)y, \pi, \tau) \geq V(\alpha g + (1 - \alpha)x, \tau, \pi) = \alpha V(g, \tau, \pi) + (1 - \alpha)u(x).$$

As $u(x) > u(y)$, then $V(f, \pi, \tau) > V(g, \tau, \pi)$. Thus (30) holds. \hfill \square

Thus given any partitions $\pi, \tau \in \Pi$, the preference $\succeq^*$ that is represented by $V^*(f) = V(f, \pi, \tau)$ on $\mathcal{H}$ satisfies all conditions of the multiple priors model in Gilboa and Schmeidler [9]. Moreover, for all $x \in X$, $V^*(x) = u(x)$. Thus Gilboa–Schmeidler’s Theorem 1 implies that there is a unique set $M(\pi, \tau) \in \mathcal{C}$ such that

$$V(f, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(f(q)) \quad \text{for all } f \in \mathcal{H}. \quad (31)$$

The combination of representations (31) and (30) implies that $\succeq$ has the required BPMEU representation

$$f \succeq g \iff \min_{q \in M(\pi f, \pi g)} u(f(q)) \geq \min_{q \in M(\pi g, \pi f)} u(g(q)) \quad (32)$$

where $u \in \mathcal{U}$ and $M : \Pi \times \Pi \rightarrow \mathcal{C}$.

Moreover, identification (32) implies additional properties for the closed and convex sets $M(\pi, \tau)$.
Lemma 13. For any \( \pi, \tau \in \Pi \),

(i) \( M(\pi, \tau) = \{ q \in \Delta : q_\pi \in M_\pi(\pi, \tau) \} \);

(ii) if \( \succeq \) is monotonic, then \( M_\pi(\pi, \tau) \subseteq M_\pi(\tau, \pi) \),

(iii) if \( \succeq \) is transitive, then \( M(\pi, \tau) = M(\pi, \pi) \),

(iv) if \( \succeq \) satisfies MT, then \( M(\pi, \tau) = M(\pi, \pi \vee \tau) \).

Proof. Fix any \( \pi, \tau \in \Pi \).

Show (i). Let \( P(\pi, \tau) = \{ q \in \Delta : q_\pi \in M_\pi(\pi, \tau) \} \).

By definition, \( M(\pi, \tau) \subseteq P(\pi, \tau) \). Conversely, take any \( p \in P(\pi, \tau) \). Then there is \( p^* \in M(\pi, \tau) \) such that \( p_\pi = p_\pi^* \). Take any \( f \in H \) and \( (\alpha, z) \in Z(f, \pi, \tau) \) such that

\[
\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \succ z \quad \text{for some } f_\pi \in H_\pi \text{ and } f_\tau \in H_\tau.
\]

It follows that

\[
\alpha u(f(p)) \geq u(f_\pi(p)) = u(f_\pi(p^*)) \geq \min_{q \in M(\pi, \tau)} u(f(q)) = V(f_\pi, \pi, \tau) > V(f_\tau, \tau, \pi) > u(z).
\]

Thus \( u(f(p)) > \frac{u(z)}{\alpha} \) for all \( (\alpha, z) \in Z(f, \pi, \tau) \). By definition (23), \( u(f(p)) \geq V(f, \pi, \tau) \). As \( p \in P(\pi, \tau) \) is arbitrary, then

\[
\min_{q \in P(\pi, \tau)} u(f(q)) \geq V(f, \pi, \tau).
\]

Yet \( V(f, \pi, \tau) \geq \min_{q \in P(\pi, \tau)} u(f(q)) \) because \( M(\pi, \tau) \subseteq P(\pi, \tau) \). Thus

\[
\min_{q \in P(\pi, \tau)} u(f(q)) = V(f, \pi, \tau).
\]

By definition, \( P(\pi, \tau) \) is convex and closed. Thus \( P(\pi, \tau) = M(\pi, \tau) \) by the uniqueness of the multiple priors representation.

Turn to (ii). Suppose that \( \succeq \) is monotonic. Take any \( f \in H_\pi \), and \( (\alpha, z) \in Z(f, \pi, \tau) \) such that

\[
\alpha f + (1 - \alpha)c_0 \geq f_\pi \succ f_\tau \succ z \quad \text{for some } f_\pi \in H_\pi \text{ and } f_\tau \in H_\tau.
\]

Let \( f'_\pi = \alpha f + (1 - \alpha)c_0 \). By (22), there is \( f'_\tau \in H_\tau \) such that

\[
f'_\pi \succeq f_\tau \succ f_\pi \succ f'_\tau \succeq z.
\]
Suppose that \( f'_\tau \succeq f'_\pi \). As \( \succeq \) is monotonic, then
\[
f'_\tau \succeq f'_\pi \succeq f_\pi.
\]
By PPT, \( f'_\tau \succeq f_\pi \). By RM, \( f_\pi \succ f'_\tau \). This contradiction implies that \( f'_\pi \succ f'_\tau \), and
\[
\alpha f + (1 - \alpha)c_0 \succeq f'_\pi \succ f'_\tau \succ z.
\]
Thus \( (\alpha, z) \in Z(f, \pi, \tau) \) and
\[
V(f, \pi, \tau) \geq V(f, \tau, \pi). \tag{33}
\]

Suppose that \( \pi = \{E_1, \ldots, E_n\} \). Interpret each element \( q_\pi \in \Delta_\pi \) as a vector
\[
(q_\pi(E_1), \ldots, q_\pi(E_n)) \in \mathbb{R}^n.
\]
Let \( A = M_\pi(\pi, \tau) \subset \mathbb{R}^n \) and \( B = M_\pi(\tau, \pi) \subset \mathbb{R}^n \). Both \( A \) and \( B \) are closed and convex sets. Suppose that \( A \setminus B \neq \emptyset \). Take \( a \in A \setminus B \). By the separation theorem, there is a vector \( b \in \mathbb{R}^n \) such that
\[
a \cdot b < \min_{c \in B} c \cdot b.
\]
By continuity, \( b \) can be taken so that \( b_i \neq b_j \) for all \( i, j \in \{1, \ldots, n\} \). Rescale \( b \) so that \( b_i \in [-1, 1] \) for all \( i \). Define an act \( g \) such that \( u(g(s)) = b_i \) for all \( i \) and \( s \in E_i \). Then \( g \in \mathcal{H}_\pi \) and
\[
V(g, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(g(q)) \leq a \cdot b < \min_{c \in B} c \cdot b = \min_{q \in M(\tau, \pi)} u(g(q)) = V(g, \tau, \pi).
\]
This contradiction with (33) shows that \( A \subseteq B \), that is, \( M_\pi(\pi, \tau) \subseteq M_\pi(\tau, \pi) \).

Turn to (iii). Suppose that \( \succeq \) is transitive. Take any \( f \in \mathcal{H} \) and \( (\alpha, z) \in Z(f, \pi, \tau) \) such that
\[
\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f_\tau \succ z \quad \text{for some } f_\pi \in \mathcal{H}_\pi \text{ and } f_\tau \in \mathcal{H}_\tau.
\]
Take \( f'_\pi \in \mathcal{H}_\pi \) such that \( f_\tau \gg f'_\pi \gg z \). By transitivity and RM, \( f_\pi \succ f'_\pi \). Thus \((\alpha, z) \in Z(f, \pi, \pi)\) and hence, \( V(f, \pi, \pi) \geq V(f, \pi, \tau) \). Similarly, if
\[
\alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f'_\pi \gg z \quad \text{for some } f_\pi, f'_\pi \in \mathcal{H}_\pi,
\]
then there is \( f_\tau \in \mathcal{H}_\tau \) such that \( \alpha f + (1 - \alpha)c_0 \succeq f_\pi \succ f_\tau \gg z \). Thus \( V(f, \pi, \tau) \geq V(f, \pi, \pi) \). The equality \( V(f, \pi, \tau) = V(f, \pi, \pi) \) for all \( f \in \mathcal{H} \) implies that \( M(\pi, \tau) = M(\pi, \pi) \).

Turn to (iv). Suppose that \( \succeq \) satisfies MT. Let \( A = M_\pi(\pi, \tau) \) and \( B = M_\pi(\pi, \pi \lor \tau) \). Suppose that \( A \neq B \). Consider two cases.
Case 1. There is \( a \in A \setminus B \). Then there is an act \( g \in \mathcal{H}_\pi \) such that

\[
\min_{q \in M(\pi,\tau)} u(g(q)) < \min_{q \in M(\pi,\pi \lor \tau)} u(g(q)).
\]

Take \( x, y, z \in X \) such that

\[
\min_{q \in M(\pi,\pi \lor \tau)} u(g(q)) > u(x) > u(y) > u(z) > \min_{q \in M(\pi,\tau)} u(g(q)).
\]

By (22), there is \( h \in \mathcal{H}_\pi \) such that \( y \gg h \gg z \). Moreover, \( h \) can be taken so that \( \pi \left( \frac{h + q}{2} \right) = \pi \lor \tau \). Take \( x' = \frac{y + x}{2} \) and \( \alpha \in (0, 0.5) \) small enough so that \( x \gg \alpha g + \alpha h + (1 - 2\alpha)x' \gg y \). Note that

\[
\pi(\alpha g + \alpha h + (1 - 2\alpha)x') = \pi \left( \frac{h + q}{2} \right) = \pi \lor \tau.
\]

Then

- \( h \succ g \) because

\[
V(h, \tau, \pi) = \min_{q \in M(\tau,\pi)} u(h(q)) > u(x) > \min_{q \in M(\pi,\tau)} u(g(q)) = V(g, \pi, \tau),
\]

- \( \alpha g + \alpha h + (1 - 2\alpha)x' \succ h \) because

\[
V(\alpha g + \alpha h + (1 - 2\alpha)x', \pi \lor \tau, \tau) > u(y) > V(h, \tau, \pi \lor \tau),
\]

- \( g \succ \alpha g + \alpha h + (1 - 2\alpha)x' \) because

\[
V(g, \pi, \pi \lor \tau) = \min_{q \in M(\pi,\pi \lor \tau)} u(g(q)) > u(x) > V(\alpha g + \alpha h + (1 - 2\alpha)x', \pi \lor \tau, \pi).
\]

However, MT prohibits the cycle \( g \succ \alpha g + \alpha h + (1 - 2\alpha)x' \succ h \succ g \).

Case 2. There is \( b \in B \setminus A \). Then there is an act \( g \in \mathcal{H}_\pi \) such that

\[
\min_{q \in M(\pi,\tau)} u(g(q)) > \min_{q \in M(\pi,\pi \lor \tau)} u(g(q)).
\]

Take \( x, y, z \in X \) such that

\[
\min_{q \in M(\pi,\tau)} u(g(q)) > u(x) > u(y) > u(z) > \min_{q \in M(\pi,\pi \lor \tau)} u(g(q)).
\]

By (22), there is \( h \in \mathcal{H}_\pi \) such that \( x \gg h \gg y \). Moreover, \( h \) can be taken so that \( \pi \left( \frac{h + q}{2} \right) = \pi \lor \tau \). Take \( x' = \frac{y + x}{2} \) and \( \alpha \in (0, 0.5) \) small enough so that \( y \gg \alpha g + \alpha h + (1 - 2\alpha)x' \gg z \). Note that

\[
\pi(\alpha g + \alpha h + (1 - 2\alpha)x') = \pi \left( \frac{h + q}{2} \right) = \pi \lor \tau.
\]

Then
• $g \succ h$ because
\[ V(g, \pi, \tau) = \min_{q \in M(\pi, \tau)} u(g(q)) > u(x) > \min_{q \in M(\tau, \pi)} u(h(q)) = V(h, \pi, \tau), \]

• $\alpha g + \alpha h + (1 - 2\alpha)x' \succ g$ because
\[ V(\alpha g + \alpha h + (1 - 2\alpha)x', \pi \lor \tau, \pi) > u(z) > V(g, \pi, \pi \lor \tau), \]

• $h \succ \alpha g + \alpha h + (1 - 2\alpha)x'$ because
\[ V(h, \tau, \pi \lor \tau) > u(y) > V(\alpha g + \alpha h + (1 - 2\alpha)x', \pi \lor \tau, \tau). \]

However, MT prohibits the cycle $h \succ \alpha g + \alpha h + (1 - 2\alpha)x' \succ g \succ h$.

The above contradictions imply that $A = B$. By (i) $M(\pi, \tau) = M(\pi, \pi \lor \tau)$. $\square$

To show the uniqueness claim in Theorem 3, suppose that $\succeq$ has another representation (32) with components $u^* \in U$ and $M^* : \Pi \times \Pi \to C$. Then $u^*$ is a plt of $u$ because both $u$ and $u^*$ represent the same ranking of lotteries. Wlog take $u^* = u$.

Suppose that $M^*(\pi, \tau) \neq M^*_\pi(\pi, \tau)$ for some $\pi, \tau \in \Pi$. The uniqueness claim in Gilboa–Schmeidler’s Theorem 1 implies that the two utility functions
\[
U(f) = \min_{q \in M^*(\pi, \tau)} u(f(q)) \\
U^*(f) = \min_{q \in M^*_\pi(\pi, \tau)} u(f(q))
\]
represent distinct rankings over acts $f \in \mathcal{H}_\pi$. Take $f \in \mathcal{H}_\pi$ such that $U(f) \neq U^*(f)$. Wlog $U(f) > U^*(f)$. Take $x, y \in X$ such that $U(f) > u(x) > u(y) > U^*(f)$. Then $f \gg x \gg y$. By (22), there is $g \in \mathcal{H}_\pi$ such that $f \gg x \gg g \gg y$. Then
\[
\min_{q \in M(\pi f, \pi g)} u(f(q)) = U(f) > u(x) > \min_{q \in M(\pi g, \pi f)} u(g(q)) \\
\min_{q \in M^*(\pi g, \pi f)} u(g(q)) > u(y) > U^*(f) = \min_{q \in M^*(\pi f, \pi g)} u(f(q)).
\]
The former inequality implies $f \succ g$, and the latter inequality implies $g \succeq f$. This contradiction establishes the equalities $M^*_\pi(\pi, \tau) = M^*_\pi(\pi, \tau)$ for all $\pi, \tau \in \Pi$.

Turn to the monotonic and transitive cases. Suppose that $\succeq$ is monotonic.
References


