Simple axioms for countably additive subjective probability

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A B S T R A C T

This paper refines Savage’s theory of subjective probability for the case of countably additive beliefs. First, I replace his continuity axioms P6 and P7 with a simple modification of Arrow’s (1970) Monotone Continuity. Second, I relax Savage’s primitives: in my framework, the class of events need not be a \( \sigma \)-algebra, and acts need not have finite or bounded range. By varying the domains of acts and events, I obtain a unique extension of preference that parallels Caratheodory’s unique extension of probability measures. Aside from subjective expected utility, I characterize exponential time discounting in a setting with continuous time and an arbitrary consumption set.

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1. Introduction

Savage (1954) derives subjective probabilities from individual choice behavior rather than from objective symmetries or empirical frequencies of events. Given a state space \( S \) with a \( \sigma \)-algebra of events \( \mathcal{A} \), he obtains a unique probability measure \( p \) on \( \mathcal{A} \) and an expected utility representation

\[
U(f) = \int_S u(f(s)) \, dp
\]

from preferences over acts — \( \mathcal{A} \)-measurable functions \( f \) that map \( S \) into a given set \( X \) of possible outcomes. In Savage’s model, the subjective belief \( p \) is finitely rather than countably additive, every act \( f \) must have either finite or bounded range and hence, every function \( u(f(s)) \) is bounded.\(^1\)

To guarantee that the belief \( p \) is countably additive, Arrow (1970) adds Monotone Continuity to Savage’s postulates P1–P7. (Monotone Continuity is originally proposed by Villegas, 1964 for preferences over events.) Arrow’s analysis still requires that the primitive domain of events should be a \( \sigma \)-algebra, and all acts should be bounded.

This paper refines the foundations for countably additive subjective probability. First, I dispense with Savage’s continuity axioms P6 and P7. To do so, I strengthen Arrow’s Monotone Continuity and impose it for nested sequences of events that have an empty (as in Arrow and Villegas) or singleton intersection. This formulation is much shorter and appears more convenient in applications than the combination of P6, P7, and Monotone Continuity.\(^2\)

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1 Savage writes on p. 82 that “problems involving unbounded utility are to be handled cautiously” and essentially excludes such problems from his analysis. He also acknowledges that the assumption that \( \mathcal{A} \) is a \( \sigma \)-algebra is essential for his proof, but notes that it is peculiar that “one should use countable unions of events in order to derive a finitely additive probability measure.”

2 Both P6 and P7 still hold in my model, but they are implied by the other axioms rather than assumed directly.

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Second, I relax the mathematical structure imposed on the primitive class of events \( \mathcal{A} \). In my framework, \( \mathcal{A} \) need not be closed under countable set operations. For example, if the state space \( S = \{0, +\infty\} \) is one-dimensional, then \( \mathcal{A} \) may be the set of all intervals \([a, b)\) for \(0 \leq a < b \leq +\infty\). By contrast, the Borel \( \sigma \)-algebra \( \sigma(\mathcal{A}) \), which is the minimal \( \sigma \)-algebra that contains \( \mathcal{A} \), cannot be readily described via combinations of intervals. More precisely, for any countable ordinal \( \gamma \), there are sets in \( \sigma(\mathcal{A}) \) that cannot be arrived at from \( \mathcal{A} \) by taking a \( \gamma \)-sequence of complements, unions, and intersections. (See Billingsley, 1995, pp. 31–32 for details.) In this sense, the class \( \sigma(\mathcal{A}) \) is substantially more complex than \( \mathcal{A} \).

Third, the domain \( \mathcal{F} \) of the preference \( \succ \) in my framework is not restricted to bounded acts, as in Savage, and may include all \( \mathcal{A} \)-measurable acts \( f \) for which the function \( u \circ f \) is Lebesgue integrable on the probability space \((S, \sigma(\mathcal{A}), p)\). For example, if \( u \) is an exponential (CARA) utility for money, then \( \mathcal{F} \) may include all acts that induce normal probability distributions over monetary payoffs (see Kreps, 1988, p. 68).

My main result shows that Savage’s four well-known axioms P1–P4 (Order, Sure-Thing Principle, Monotonicity, Comparative Probability) and my strong version of Monotone Continuity are sufficient for the preference \( \succ \) to have an expected utility representation (1). Note that the integrability of all functions \( u \circ f \) is not assumed exogenously, but is derived together with the utility index \( u \) and the countably additive subjective belief \( p \) from the five axioms imposed on the preference \( \succ \).

By varying the domain \( \mathcal{F} \), the preference \( \succ \) can be uniquely extended from the class of all \( \mathcal{A} \)-measurable acts with finite range to the class of all \( \sigma(\mathcal{A}) \)-measurable integrable acts. This extension (Corollary 2) parallels Caratheodory’s extension of countably additive probability measures.

Aside from subjective expected utility, I characterize the exponential time discounting model where acts \( f \) are interpreted as deterministic consumption programs defined over the continuous range of time \([0, +\infty)\). The preference \( \succ \) over such programs is represented by

\[
U(f) = \int_0^\infty e^{-\gamma t}u(f(t)) \, dt
\]

for some positive discount rate \( \gamma > 0 \). Here the function \( u(f(t)) \) is not assumed to be bounded and, unlike Koopmans’ (1960) classic model of time discounting, the set of payoffs \( X \) need not be connected. The freedom in the specification of \( X \) can be useful in modeling preferences over durable indivisible goods, like houses or automobiles, for which connectedness is problematic.

There are several related models in the literature. Kopylov (2007) extends the theory of subjective probability from \( \sigma \)-algebras to “smaller” domains, such as algebras, \( \lambda \)-systems, and more general structures called mosaics. This paper invokes Kopylov’s results, but uses another continuity axiom, another domain of preference, and delivers a countably additive probability measure \( p \).

Wakker (1993) characterizes an expected utility representation for unbounded acts with a finitely additive subjective probability measure. His primitives and assumptions are much different from the ones adopted here. In particular, one of the conditions (step-equivalence) that he adds to Savage’s P1–P7 requires that for any act \( f \) in the domain \( \mathcal{F} \), there is a simple (i.e. finite-valued) act \( g \) such that \( f \sim g \). This condition need not hold in my model. Wakker briefly discusses countable additivity and mentions that in this case, ‘several parts in the analysis of [his] paper could be abbreviated and simplified’. My results formalize his intuition.

Bleichrodt et al. (2008) extend Koopmans’s model of time discounting for unbounded programs with discrete time. These authors take \( X \) to be connected, and assume constant-equivalence that is similar to step-equivalence in Wakker’s analysis.

2. Preliminaries

Given are

(i) a set \( X = \{x, y, \ldots\} \) of deterministic outcomes,

(ii) a set \( S = \{s, t, \ldots\} \) of states of the world,

(iii) a family \( \mathcal{A} = \{A, B, \ldots\} \subset 2^S \) of subsets of \( S \) called events,

(iv) a family \( \mathcal{F} = \{f, g, \ldots\} \subset X^S \) of functions \( f: S \rightarrow X \) called acts,

(v) a binary relation \( \succ \) on \( \mathcal{F} \).

Each act \( f \in \mathcal{F} \) represents a physical action with state-contingent outcomes \( f(s) \). The relation \( \succ \) describes a decision maker’s weak preference over such actions. The strict part \( \succ^* \) of this preference is assumed throughout to be non-empty. This non-degeneracy assumption corresponds to P5 in Savage’s analysis.

Identify outcomes \( x \in X \) with the corresponding constant functions. For any event \( A \) and acts \( f, g \in \mathcal{F} \), let \( f \upharpoonright A \) denote the composite function that yields \( f(s) \) if \( s \in A \) and \( g(s) \) if \( s \notin A \). Write \( f \succeq g \) if \( f(s) \succeq g(s) \) for all \( s \in S \).

Impose the following structural assumptions on the primitives of the model.

(I) \( \mathcal{A} \) is an algebra, that is, \( \mathcal{A} \) is closed under taking complements and finite unions.

(II) \( \mathcal{A} \) is countably separated, that is, \( \mathcal{A} \) contains a countable collection of events \( \mathcal{G} \subset \mathcal{A} \) such that for any \( s, s' \in S \), there is \( E \in \mathcal{G} \) such that \( s \in E \) but \( s' \notin E \).
(III) $xAf \in \mathcal{F}$ for all $f \in \mathcal{F}, x \in X$, and $A \in \mathcal{A}$.

(IV) $\{s \in S : f(s) \succ x\} \in \mathcal{A}$ and $\{s \in S : f(s) \prec x\} \in \mathcal{A}$ for all $f \in \mathcal{F}$ and $x \in X$.

These assumptions hold automatically in many natural settings. For example, $\mathcal{A}$ is countably separated if

- $S \subset \mathbb{R}^n$ or $S \subset \mathbb{R}^\infty$, and $\mathcal{A}$ is any algebra that contains sets $\{s \in S : a \leq s_i\}$ for all $i$ and all $a \in \mathbb{R}$; if needed, $a$ can be assumed to be rational, or to have a finite decimal expansion;
- $S = \prod_{i=1}^\infty \Omega_i$ is a product of finite or countable sets $\Omega_i$, and $\mathcal{A}$ is any algebra that contains sets $\{s \in S : s_i = \omega_i\}$ for all $i$ and all $\omega_i \in \Omega_i$;
- $\mathcal{S}$ is endowed with a Hausdorff topology, and $\mathcal{A}$ is any algebra that contains a countable base for this topology.

The definition of countable separability for $\sigma$-algebras is due to Mackey (1957). Note though that $\mathcal{A}$ need not be a $\sigma$-algebra in any of the above examples.

Next, the measurability conditions (III) and (IV) hold if the domain $\mathcal{F}$ has one of the following specifications.

- $\mathcal{F}$ is the set $\mathcal{F}(\mathcal{A})$ of all $\mathcal{A}$-measurable functions $f : S \to X$ that have finite range. Elements of $\mathcal{F}(\mathcal{A})$ are called simple acts. Note that $\mathcal{F}(\mathcal{A})$ is the smallest domain of acts that satisfies condition (III) for the given algebra $\mathcal{A}$. Therefore, if $\mathcal{A}$ is a universal $\sigma$-algebra, such as the power set $2^S$, then the set $\mathcal{F}$ in my framework must contain Savage’s standard domain of simple acts.
- $\mathcal{F}$ is the set $\mathcal{M}(\mathcal{A})$ of all functions $f : S \to X$ such that $\{s \in S : f(s) \succ x\} \in \mathcal{A}$ and $\{s \in S : f(s) \prec x\} \in \mathcal{A}$ for all $x \in X$. In contrast with simple acts, the specification of the domain $\mathcal{M}(\mathcal{A})$ depends on the ranking of deterministic outcomes in $X$. This ranking, written as $\succ$, needs to be observed prior to its extension $\succsim$ to the entire $\mathcal{M}(\mathcal{A})$. Note that $\mathcal{M}(\mathcal{A})$ is the largest domain of acts that satisfies condition (IV) for the given $\mathcal{A}$ and $\succsim$.
- $\mathcal{F}$ is the set $\mathcal{B}(\mathcal{A})$ of all functions $f \in \mathcal{M}(\mathcal{A})$ such that $x \succeq f \succeq y$ for some $x, y \in X$. Elements of $\mathcal{B}(\mathcal{A})$ are called bounded acts.

Fix a utility index $u : X \to \mathbb{R}$ that represents the ranking $\succeq_0$ of $X$, and a countably additive probability measure $p$ on the algebra $\mathcal{A}$. Let $\mathcal{F}$ be the set $\mathcal{L}(\mathcal{A}, u, p)$ of all functions $f \in \mathcal{M}(\mathcal{A})$ such that $u \circ f$ is Lebesgue integrable on the probability space $(S, \mathcal{A}, p)$. Then

$$\mathcal{F}(\mathcal{A}) \subset \mathcal{B}(\mathcal{A}) \subset \mathcal{L}(\mathcal{A}, u, p) \subset \mathcal{M}(\mathcal{A}).$$

Note that if $u$ is a positive monotonic transformation of $u$, then the domain $\mathcal{L}(\mathcal{A}, v, p)$ need not equal $\mathcal{L}(\mathcal{A}, u, p)$.

Let $\sigma(\mathcal{A})$ be the minimal $\sigma$-algebra that contains $\mathcal{A}$. Let $\mathcal{P}$ be the set of all countably additive probability measures on the $\sigma$-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$. As $\mathcal{A}$ is countably separated, then $(s) \in \sigma(\mathcal{A})$ for all $s \in S$ because $(s) = \bigcap_{E \in \mathcal{G} : s \in E} E$.

Let $\mathcal{P}_0$ be the set of all probability measures $p \in \mathcal{P}$ such that

$$p(|s|) = 0 \quad \text{for all} \quad s \in S. \quad (3)$$

Note that every probability measure $p \in \mathcal{P}_0$ must be convex ranged, that is, for all $B \in \sigma(\mathcal{A})$,

$$\{p(A) : A \in \sigma(\mathcal{A}), A \subset B\} = [0, p(B)].$$

(See Lemma 8.) Thus, given that $\mathcal{A}$ is countably separated and $p$ is countably additive, condition (3) is equivalent to convex-rangedness.

3. Axioms and main representation result

The following four axioms are direct extensions of Savage’s postulates P1–P4.

**Axiom 1 (Order)**. $\succ$ is complete and transitive.

**Axiom 2 (Sure-Thing Principle)**. For all acts $f, g, h, h' \in \mathcal{F}$ and events $A \in \mathcal{A}$,

$$fAh \succ gAh \Rightarrow fAh' \succ gAh'. \quad (STP)$$

This axiom (STP for short) asserts that the ranking of any acts $f$ and $g$ conditioned on any event $A$ should be unaffected by outcomes that are obtained if $A$ does not occur.

**Axiom 3 (Monotonicity)**. For all $f, g \in \mathcal{F}$, if $f \succeq g$, then $f \succeq g$.

Monotonicity is intuitive if the ranking of outcomes in $X$ is state-invariant.

**Axiom 4 (Comparative Probability)**. For all events $A, B \in \mathcal{A}$ and outcomes $x \succ y$ and $x' \succ y'$,

$$xAy \succ xBy \Rightarrow x'Ay' \succ x'By'. \quad (CP)$$
Comparative Probability asserts that the preference to bet on an event $A$ rather than on $B$ should be unaffected by the stakes in these bets.

Given a sequence of events $A_1, A_2, \ldots \in \mathcal{A}$, write $A_i \succ \emptyset$ if $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{i=1}^{\infty} A_i$ is either empty or singleton.

**Axiom 5 (Strong Monotone Continuity).** For all acts $f, g \in \mathcal{F}$, outcomes $x \in X$, and events $A_1, A_2, \ldots$ such that $A_i \succ \emptyset$, if $f \succ \alpha A g$ or $\alpha A f \succ \beta g$ for all $i$, then $f \succ \beta g$.

This axiom (SMC for short) strengthens the monotone continuity conditions in Villegas (1964) and Arrow (1970), where $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Moreover, SMC requires that if $\{s\} \in \mathcal{A}$, then

$$f \succ x \| s \| f$$

for all outcomes $x \in X$ and acts $f \in \mathcal{F}$. The nullity of singleton events is intuitive if the description of each state of the world includes an infinite sequence of stochastic data, such as coin flips, sun spots, market prices, etc. Conversely, given that all singleton sets $\{s\}$ belong to the algebra $\mathcal{A}$, SMC follows from Arrow’s Monotone Continuity and the nullity condition (4). Note that if $S$ is finite, then some singleton events are not null and hence, $\succ$ violates SMC. Moreover, if $S$ is countable, then $P_0$ is empty and hence, Axioms 1–5 are inconsistent with representation (5). By contrast, Savage’s finitely additive model can be formulated on a countable state space.

Let $\mathcal{U}$ be the set of all non-constant functions $u : X \to \mathbb{R}$.

**Theorem 1.** $\succ$ satisfies Axioms 1–5 if and only if $\succ$ is represented by

$$U(f) = \int_{\mathcal{A}} u(f(\{s\})) dp,$$

where $u \in \mathcal{U}$ and $p \in P_0$ are such that $\mathcal{F} \subset \mathcal{L}(\mathcal{A}, u, p)$.

Moreover, $\succ$ has another representation (5) with components $u' \in \mathcal{U}$ and $p' \in P$ if and only if $p' = p$ and $u' = au + \beta$ for some $a > 0$ and $\beta \in \mathbb{R}$.

The hard part in the proof of this theorem is to construct the utility representation (5) from Axioms 1–5. This construction proceeds in two big steps.

1. Invoke Kopylov’s (2007) extension of Savage’s theory to obtain the expected utility representation (5) on the subdomain $\mathcal{F}(\mathcal{A}) \subset \mathcal{F}$ of simple acts. Savage’s result does not apply here because $\mathcal{A}$ need not be a $\sigma$-algebra.

2. Use SMC to show that the expected utility function (5) is well-defined on the entire $\mathcal{F}$, and represents the preference $\succ$.

All details are relegated to Appendix A.

As in Savage, the probability measure $p$ in the expected utility representation (5) is subjective, that is, $p$ is derived from preferences over acts rather than from objective symmetry or statistical data.

**Theorem 1** differs from Savage’s result in several aspects. First, the probability measure $p$ in representation (5) is countably rather than finitely additive. Countable additivity is necessary for the law of large numbers, the central limit theorem, the Radon-Nikodym and de Finetti Theorems, and many other important mathematical results. Countable additivity is also necessary to avoid money pumps (Seidenfeld and Schervish, 1983), to satisfy conglomerability (Dubins, 1975), and strict stochastic dominance (Wakker, 1993).

Second, the continuity condition (SMC) replaces Savage’s P6 (Small Event Continuity) and Arrow’s Monotone Continuity, which are both required in the standard construction of countably additive subjective probabilities. P6 assumes that for any acts $f \succ g$ and any outcome $x$, there exists a partition of the universal event $S$ into “small” events $S_i$ such that $x S_i \succ g$ and $f \succ x S_i g$ for all $i = 1, \ldots, m$. This existence condition seems less practical than SMC. Unlike SMC, P6 need not be preserved when the primitive class of acts $A$ is reduced to a smaller algebra (or $\sigma$-algebra), $A' \subset A$. Moreover, SMC should be more appealing to probability theorists who define continuity of measures via nested sequences of events rather than via partitions of the state space. It should be emphasized though that, given all other assumptions of **Theorem 1**, SMC implies P6. (See Lemma 6 in Appendix A.)

Third, the primitive domain of events $\mathcal{A}$ in **Theorem 1** is assumed to be an algebra rather than a $\sigma$-algebra or, a fortiori, the power set $2^X$ as in Savage. This relaxation allows to specify all events in $\mathcal{A}$ explicitly. Indeed, the minimal algebra $\mathcal{A}$ that

1. This implication does not hold in general. To show this claim, let $X = \{x, y\}$ and $S = \prod_{i=1}^{\infty} (0, 1)$. Take a five-element subset $S \subset S$. Following Kraft et al. (1959), take a qualitative probability $\succ$, over subsets of $S$, that cannot be represented by a probability measure. Let $\mathcal{A}$ be the algebra of cylinders—sets that have the form

$$\{s \in S : (s_1, \ldots, s_n) \in A\}$$

for some finite $n$ and some $A \in \prod_{i=1}^{\infty} (0, 1)$. Note that $\mathcal{A}$ is countably separated, but does not contain singleton sets. For all events $A, B \in \mathcal{A}$, let $x A y \succ x B y$ if and only if $A \cap S \succ B \cap S$. Then the preference $\succ$ satisfies Axioms 1–4, Arrow’s Monotone Continuity, the nullity condition (4), but violates SMC and cannot be represented by subjective probabilities.

4. See Billingsley (1995, Theorems 22.1 and 27.1). The limit laws for finite additive probabilities, such as Purves and Sudderth (1976), are less powerful and more cumbersome in applications.
contains a given family of basic events \( \mathcal{G} \subset 2^S \) (e.g. the family of intervals on the real line) is the collection of all finite disjoint unions of sets that have the form \( \cap_{i=1}^n A_i \), where \( A_i \in \mathcal{G} \) or \( A_i^c \in \mathcal{G} \) for all \( i = 1, \ldots, n \). By contrast, events in the minimal \( \sigma \)–algebra that contains \( \mathcal{G} \) (e.g. the Borel sets on the real line) do not have any such explicit specification. (See Billingsley, 1995, pp. 31–32 and Problem 2.5 for details.)

Moreover, the weaker mathematical structure imposed on \( \mathcal{A} \) allows all non-empty events in \( \mathcal{A} \) to be non-null. For example, if \( \mathcal{A} \) is generated by intervals \([a, b]\) in \( S = [0, 1] \), and \( p \) is the uniform distribution, then \( p(A) > 0 \) for all non-empty \( A \in \mathcal{A} \). Therefore, for all \( A, B \in \mathcal{A} \) such \( A \supset B \), \( A \) may be subjectively strictly more likely than \( B \). Some authors, most prominently Keynes (1921) and Koopman (1940), find such strict Monotonicity to be normatively desirable for comparative likelihood relations.\(^5\) Note that in the absence of non-empty null events, one can also derive a unique subjective belief conditional on any non-empty event \( C \in \mathcal{A} \) by taking

\[
p_C(A) = \frac{p(A \cap C)}{p(C)}
\]

for all \( A \in \mathcal{A} \). The conditional belief \( p_C \) inherits countable additivity from \( p \).

Finally, the domain of preference \( \mathcal{F} \) in Theorem 1 is not restricted to simple or bounded acts (as in Savage) and may contain any acts \( f \) such that the function \( u \circ f \) is \( \mathcal{A} \)–measurable and Lebesgue integrable on the probability space \((S, \sigma(A), p)\). Moreover, the integrability of all acts in \( \mathcal{F} \) is not assumed exogenously, but derived from axioms imposed on the preference \( \succ \). It follows that the preference \( \succ \) satisfies Axioms 1–5 on the largest admissible domain \( \mathcal{F} = \mathcal{M}(A) \) if and only if \( \succ \) has representation (5) with a bounded utility index \( u \in \mathcal{U} \).

By varying the domain \( \mathcal{F} \), one can extend the preference \( \succ \) uniquely from the domain of \( \mathcal{A} \)–measurable simple acts to the much bigger domain of all \( \sigma(A) \)–measurable acts that are integrable on the probability space \((S, \sigma(A), p)\).

**Corollary 2.** If the preference \( \succ \) in a tuple \((S, X, A, \mathcal{F}(A), \succ)\) satisfies Axioms 1–5, then it has a unique extension \( \succ_s \) that satisfies Axioms 1–5 in a tuple \((S, X, A_s, \mathcal{F}_s, \succ_s)\), where \( A_s = \sigma(A) \) and \( \mathcal{F}_s = \mathcal{L}(\sigma(A), u, p) \).

Indeed, if the extension \( \succ_s \) on \( \mathcal{F}_s \) is represented by an expected utility function with components \( u, p \in \mathcal{U} \) and \( p \in \mathcal{P}_0 \), then the same function represents \( \succ \) on \( \mathcal{F} \). Therefore the preference \( \succ_s \) is uniquely determined by \( \succ \). This extension of the preference \( \succ \) parallels the Caratheodory extension of countably additive probability measures and the construction of the Lebesgue integral. Analogously to Lebesgue, one can strengthen Corollary 2 further by taking \( A_s \) to be the completion of the \( \sigma \)–algebra \( \sigma(A) \) with respect to the probability measure \( p \).

4. One-dimensional state space and time discounting

To obtain more specific applications of Theorem 1, suppose that \( S = [0, +\infty) \). For example, the state \( s \) can be defined as an asset price or a physical variable, such as temperature, location, velocity, or time. Let

\[
I = \{(a, b) : 0 \leq a \leq b \leq +\infty\}
\]

be the family of all semi-open intervals in \( S \). This family is not an algebra because it is not closed under unions. Instead, \( I \) has a weaker mathematical structure called a *semialgebra*. (See Aliprantis and Border, 1999, p. 128.)

Let \( \mathcal{F}(I) \) be the set of all functions \( f:S \to X \) such that \( f \) is constant on every interval \([a_{i-1}, a_i)\) for some \( 0 = a_0 < a_1 < \cdots < a_{n-1} < a_n = +\infty \). Then \( \mathcal{F}(I) \) equals to the set \( \mathcal{F}(\mathcal{A}(I)) \) of all simple acts that are measurable with respect to the minimal algebra \( \mathcal{A}(I) \) that contains \( I \). This minimal algebra is the collection of all finite disjoint unions of intervals in \( I \).

In general, Theorem 1 does not hold if the tuple of primitives \((S, X, A, \mathcal{F}, \succ)\) satisfies the structural assumptions (II), (III), and (IV), but \( A \) is an arbitrary semialgebra rather than an algebra. Yet Theorem 1 is preserved in the special case when \( A = I \).

**Corollary 3.** Let \( A = I \) and \( \mathcal{F} = \mathcal{F}(I) \). Then \( \succ \) satisfies Axioms 1–5 if and only if \( \succ \) has the expected utility representation (5).

In this case, Comparative Probability requires that

\[
x(a, b) y \succ x(c, d) y \iff x(a, b) y' \succ x(c, d) y'
\]

for all \( 0 \leq a, c, d \leq +\infty \) and outcomes \( x > y \) and \( x' > y' \). SMC takes a simpler form as well because a sequence of nested events \([a_1, b_1) \supset [a_2, b_2) \supset \cdots \) has an empty or singleton intersection if and only if \( 0 \leq \lim a_n = \lim b_n \leq +\infty \).

Aside from choice under uncertainty, this framework can be used to model preferences over deterministic consumption streams with continuous time. In this case, elements of \( X \) are interpreted as goods that the decision maker may consume at any time moment \( t \geq 0 \). Unlike the classic characterization of time discounting in Koopmans (1960), \( X \) is not assumed to be connected or to have any other topological structure. This generality can accommodate bundles of durable goods for which connectedness is problematic.

Each act \( f \in \mathcal{F} \) is interpreted as a consumption program for all future moments of time \( t \geq 0 \). The motivation for Axioms 1–5 needs to be modified accordingly. STP becomes a version of time *separability*. Monotonicity asserts that the ranking of

\(^5\) Savage (1954, p. 39) observes that his model contradicts strict Monotonicity, but finds this property to be “entirely a matter of taste, conditioned on mathematical experience.”
consumption goods is preserved over time. Condition (6) requires that the decision maker’s choice to improve consumption from \( y \) to \( x \) at time period \([a, b]\) rather than at time period \([c, d]\) should reflect exclusively her discount rates for the two periods. SMC requires roughly that the value of any given consumption \( x \) at time periods \([a, b]\) should converge to zero if these periods reduce to a single moment in time (i.e. \( \lim a_i = \lim b_i < +\infty \)), or if the delay of consumption goes to infinity (i.e. \( \lim a_i = +\infty \)).

Besides Axioms 1–5, consider

**Axiom 6 (Stationarity).** For all \( x, y \in X \) and \( a, b, c, d, t \in [0, +\infty) \),

\[
x[a, b]y \succ x[c, d]y \iff x[a + t, b + t]y \succ x[c + t, d + t]y.
\]

Stationarity asserts that the relative discounting of consumption at time periods \([a, b]\) and \([c, d]\) should be preserved if both periods are delayed by any \( t \geq 0 \).

**Corollary 4.** Let \( A = \mathcal{I} \) and \( F = \mathcal{F}(\mathcal{I}) \). Then \( \succ \) satisfies Axioms 1–6 if and only if it is represented by

\[
U(f) = \int_0^\infty e^{-\gamma t} u(f(t)) \, dt
\]

for some \( \gamma > 0 \) and \( u \in \mathcal{U} \).

Moreover, the discounting rate \( \gamma \) is unique, and the utility index \( u \in \mathcal{U} \) is unique up to a positive linear transformation.

As in the expected utility model, one may desire to extend representation (7) to a domain of acts larger than \( \mathcal{F}(\mathcal{I}) \). For instance, macroeconomic applications may use consumption streams with unbounded functions \( u \circ f \). To extend the domain of the preference \( \succ \), one can apply Theorem 1. For example, one can take \( A = \sigma(\mathcal{I}) \) equal to the Borel \( \sigma \)-algebra on \([0, +\infty) \) and \( F = \mathcal{C}(\mathcal{A}, u, \gamma) \) equal to the class of all functions \( f \) such that \( u \circ f \) is a Borel function, and \( e^{-\gamma t} u(f(t)) \) is Lebesgue integrable on \([0, \infty) \). Then the same Axioms 1–6 for the new tuple of primitives \((S, X, A, F, \succ)\) will characterize the same utility representation (7).

**Appendix A. Proofs**

Fix any utility index \( u \in \mathcal{U} \) and probability measure \( p \in \mathcal{P}_0 \). Suppose that \( F \subseteq \mathcal{L}(\mathcal{A}, u, p) \), and the preference \( \succ \) is represented by the expected utility (5). The verification of Axioms 1–4 is standard. Turn to SMC. Take any acts \( f, g \in F \), an outcome \( x \in X \), and events \( A_1, A_2, \ldots \in \mathcal{A} \) such that \( A_i \uparrow \emptyset \). Suppose that \( xA_1 \succ g \) or \( f \succ xA_1g \) for all \( i \). For every \( A \in \sigma(\mathcal{A}) \), let

\[
\begin{align*}
\nu_f(A) &= -u(x) \cdot p(A) + \int_A u(f(s)) \, dp, \\
\nu_g(A) &= u(x) \cdot p(A) - \int_A u(g(s)) \, dp.
\end{align*}
\]

The set functions \( \nu_f \) and \( \nu_g \) are countably additive (Billingsley, 1995, Theorem 16.9). Thus they satisfy \( \lim \nu_f(A_i) = \nu_f(\cap A_i) = 0 \) and \( \lim \nu_g(A_i) = \nu_g(\cap A_i) = 0 \) because \( p(\{s\}) = 0 \) for all \( s \in S \). For any \( \varepsilon > 0 \), there is \( i \) such that \( |\nu_f(A_i)| < \varepsilon \) and \( |\nu_g(A_i)| < \varepsilon \). Then

\[
U(f) - U(g) = \begin{cases} 
U(xA_1f) - U(xAg) + \nu_f(A_i) \geq \nu_g(A_i) > -\varepsilon \text{ if } xA_1f \succ g, \\
U(f) - U(xA_1g) + \nu_g(A_i) \geq \nu_f(A_i) > -\varepsilon \text{ if } f \succ xA_1g.
\end{cases}
\]

As \( \varepsilon \) is arbitrary, then \( U(f) \geq U(g) \) and hence, \( f \succ g \). Thus, SMC holds.

Turn to sufficiency. Suppose that the preference \( \succ \) satisfies Axioms 1–5. Construct the required expected utility representation (5).

Say that a collection of events \( B \subseteq \mathcal{A} \) is union-closed if \( B \cup B' \in B \) for all \( B, B' \in B \).

**Lemma 5.** If \( B \subseteq \mathcal{A} \) is union-closed, then for any event \( A_1 \in \mathcal{A} \setminus B \), there exists a sequence of events \( A_2, A_3, \ldots \in A \setminus B \) such that \( A_i \uparrow \emptyset \).

**Proof.** Fix any event \( A_1 \in \mathcal{A} \setminus B \). Take a countable class of events \( G = \{E_1, E_2, \ldots \} \) that separates states in \( S \). Let \( A_1^1 = A_1 \). For any \( i = 2, 3, \ldots \) and \( k = 1, \ldots, 2^{i-1} \), let

\[
A_{2k-1}^i = A_{k-1}^i \cap E_{i-1}
\]

\[
A_{2k}^i = A_{k}^i \setminus E_{i-1}.
\]

By construction, \( A_{k-1}^i = A_{2k-1}^i \cup A_{2k}^i \) for all \( i \) and \( k \). As \( B \) is union-closed and \( A_{i-1}^i \in \mathcal{A} \setminus B \), then either \( A_{2k-1}^i \in \mathcal{A} \setminus B \) or \( A_{2k}^i \in \mathcal{A} \setminus B \). By induction, there is a nested sequence of events \( A_1 \supset A_{n(2)}^1 \supset A_{n(3)}^2 \supset \cdots \) such that \( A_{n(i)}^i \in \mathcal{A} \setminus B \) for all \( i \).
Show that $A^{l}_{n(i)} \sim \emptyset$. Take any two distinct states $s \neq s'$. Then there is $E_{i} \in \mathcal{G}$ such that $s \in E_{i}$ and $s' \not\in E_{i}$. If $n(i+1)$ is odd, then $s' \notin A^{l}_{n(i+1)} \subset E_{i}$. If $n(i+1)$ is even, then $s \notin A^{l}_{n(i+1)} \subset S \setminus E_{i}$. Therefore, the intersection $\bigcap_{i=1}^{\infty} A^{l}_{n(i)}$ cannot contain both states $s$ and $s'$. It follows that this intersection must be empty or singleton. □

**Lemma 6.** There exist $u \in \mathcal{U}$ and $p \in \mathcal{P}_{0}$ such that $\succ$ is represented on the domain $\mathcal{F}(A) \subset \mathcal{F}$ of simple acts by the expected utility

$$U(f) = \int_{S} u(f(s)) \, dp$$

where $u \in \mathcal{U}$ and $p \in \mathcal{P}_{0}$. In this representation, $p$ is unique, and $u$ is unique up to a positive linear transformation.

**Proof.** Show first that $\succ$ satisfies Savage’s P1–P6 on $\mathcal{F}(A)$. P1 is Order. P2 is STP. P4 is Comparative Probability. Turn to P3. Take any outcomes $x \succ y$ and any event $A \in \mathcal{A}$. By Monotonicity, $xAy \succ y$. Therefore, there are two possible cases.

- $xAy \sim y$. Then by STP, $xAh \sim yAh$ for all acts $h \in \mathcal{F}(A)$. Thus the event $A$ is null in Savage’s terminology.
- $xAy \succ y = x \not\sim y$. Take any outcomes $x', y' \in X$. If $x' \succ y'$, then by Comparative Probability, $x'Ay' \succ y' \equiv x' \not\sim y'$. If $y' \succ x'$, then by Monotonicity, $y' \succ x'Ay'$. It follows that $x' \succ y' \equiv x'Ay' \succ y'$. Then by STP, for all acts $h \in \mathcal{F}(A)$,

$$x' \succ y' \iff x'Ay' \succ y' \iff xAh \succ y'Ah.$$

Thus P3 holds for non-null events as well.

Turn to P6. Take any outcome $x \in X$ and any acts $f, g \in \mathcal{F}(A)$ such that $f \succ g$. Let $B$ be the collection of all events $B$ that can be partitioned into finitely many disjoint events $B_{1}, \ldots, B_{n}$ such that $xB_{i} \not\subseteq f$ and $xB_{i} \not\subseteq g$ for all $i$. Then $B$ is union-closed. Suppose that $S \notin \mathcal{A} \setminus B$. Let $A_{1} = S$. By Lemma 5, there exists a sequence of events $A_{2}, A_{3}, \ldots$ such that $A_{i} \not\sim \emptyset$ and $A_{i} \notin \mathcal{A} \setminus B$ for all $i$. Then $g \succ xAf$ or $xAg \succ f$ must hold. As $A_{i} \not\sim \emptyset$, then by SMC, $g \succ f$ rather than $f \succ g$. This contradiction shows that $S \notin B$. Thus P6 holds.

Invoke Kopylov’s (2007, Theorem 3.1) to conclude that there exists a utility index $u \in \mathcal{U}$ and a finitely additive probability $p$ on the algebra $\mathcal{A}$ such that the expected utility (8) represents $\succ$ on $\mathcal{F}(A)$. Kopylov’s result implies also that for any $\alpha > 0$, there exists $A \in \mathcal{A}$ such that $0 < p(A) < \alpha$.

Take any nested sequence of events $A_{1}, A_{2}, \ldots$ such that $A_{i} \not\sim \emptyset$. Suppose that $\lim_{i} p(A_{i}) > 0$. Take $A \in \mathcal{A}$ such that $\lim_{i} p(A_{i}) > p(A) > 0$. Then for any $x \succ y$, $xAy \succ xAy$. By SMC, $y \succ xAy$, which contradicts $p(A) > 0$. This contradiction shows that

$$A_{i} \not\sim \emptyset \Rightarrow \lim_{i} p(A_{i}) = 0.$$  

(9)

Thus, $p$ is countably additive on the algebra $\mathcal{A}$. By Caratheodory’s theorem (see Billingsley, 1995, Theorem 3.1), $p$ has a countably additive extension to the $\sigma$-algebra $\sigma(\mathcal{A})$. By (9), this extension satisfies $p(s) = 0$ for all $s \in S$.

Kopylov’s (2007, Theorem 3.1) asserts that $p$ is uniquely defined on the algebra $\mathcal{A}$, and the utility index $u \in \mathcal{U}$ is unique up to a positive linear transformation. Caratheodory’s theorem asserts that the countably additive extension of $p$ to the $\sigma$-algebra $\sigma(\mathcal{A})$ is also unique. □

The following lemma shows that the expected utility function $U$ is well-defined on the entire domain $\mathcal{F}$.

**Lemma 7.** Every act $f \in \mathcal{F}$ has a finite Lebesgue integral

$$U(f) = \int_{S} u(f(s)) \, dp$$

on the probability space $(S, \sigma(\mathcal{A}), p)$. Moreover, if $f$ is bounded, then

$$U(f) = \sup_{h \in \mathcal{F}(A) : f \succeq h} U(h) = \inf_{h \in \mathcal{F}(A) : f \preceq h} U(h).$$

(10)

**Proof.** Take any bounded act $f \in \mathcal{F}$ such that $x \succeq f \succeq z$ for some $x, z \in X$. Without loss of generality, let $u(z) = 0$. Take any $n = 1, 2, \ldots$. For all $m = 1, 2, \ldots$, let

$$A_{m} = \left\{ s \in S : \frac{m - 1}{n} \leq u(f(s)) < \frac{m}{n} \right\} .$$

Then $A_{m} \in \sigma(\mathcal{A})$. As $p$ is countably additive on $\sigma(\mathcal{A})$, then for all $m > 1$ such that $A_{m} \not\sim \emptyset$, there exists $y_{m} \in X$ such that $(m - 1/n) \leq u(y_{m}) < m/n$, and

$$p \left( \left\{ s \in S : u(y_{m}) > u(f(s)) \geq \frac{m - 1}{n} \right\} \right) < \frac{1}{m2^{m}}.$$
Let $B_m = \{ s \in A_m : u(y_m) > u(f(s)) \geq (m - 1/n) \}$. If $A_m = \emptyset$, then let $y_m = z$ and $B_m = \emptyset$. As $f$ is bounded, then there are finitely many non-empty sets $A_m$ and hence,

$$Y_n = \{ y_m : m = 1, 2, \ldots \}$$

is finite. For all $s \in S$, let

$$h_n(s) = \arg \max_{y \in Y_n(f(s))} u(y).$$

Then $h_n$ is an $A$-measurable simple act such that $f \geq h_n$. Moreover,

$$U(f) - U(h_n) = \sum_{m=1}^{\infty} \int_{A_m \backslash B_m} u(f(s)) - u(h_n(s)) \, dp = \sum_{m=1}^{\infty} \left[ \int_{A_m \backslash B_m} u(f(s)) - u(h_n(s)) \, dp + \int_{B_m} u(f(s)) - u(h_n(s)) \, dp \right] \leq \sum_{m=1}^{\infty} \left[ \frac{1}{n} p(A_m) + \frac{m}{n} \frac{1}{m^2} \right] = \frac{2}{n}.$$

As $n$ is arbitrary, then

$$U(f) = \sup_{h \in \mathcal{A} : f \geq h} U(h).$$

The equality $U(f) = \inf_{h \in \mathcal{A} : f \geq h} U(h)$ is analogous. (One can adapt the above argument by replacing $\succ$ and $u$ with $\preceq$ and $-u$ respectively.)

Next take any $f \in \mathcal{F}$ and $x \in X$. Let $A = \{ s \in S : f(s) > z \}$. Let $f^* = f z_1$ and $f_2 = zAf$. Without loss of generality, let $u(z) = 0$. As the function $u \circ f^*$ is non-negative, then the Lebesgue integral $\int_S u_1(f^*(s)) \, dp$ is either finite, or diverges to $+\infty$. Assume that $\int f_1 u_1(f^*(s)) \, dp = +\infty$. Then the function $u \circ f^*$ is unbounded and hence, there exist $x_0, x_1, x_2 \in X$ such that $u(x_{n+1}) > u(x_0) > u(x_1) > u(x_2) > \ldots$ Let $E_n = \{ s \in S : f_1(s) > x_0 \}$. Then $E_n \rightarrow \emptyset$. Take $k$ such that $p(E_k) < 1$. By STP, $f_1^* E_k x_{k+1} = f_1^* E_k x_k$ because $x_k E_k x_{k+1} \rightarrow x_k$.

As $f_1^* E_k x_{k+1} \geq f_1^*$, then $f_1^* E_k x_{k+1} \rightarrow f_1$. By SMC, there exists $m > k$ such that $x_m E_m (f_1^* E_k x_{k+1}) > f_1$. As $u(x_m) \geq u(x_0) = u(x_1)$, then $x_m \ni f_1$. Note that the acts $x_0, x_1, \ldots$ are bounded for all $n$. By Levi’s monotone convergence theorem and by (10),

$$U(f^*) = \lim_{n \rightarrow \infty} U(x_0 E_n f^*) = \sup_{h \in \mathcal{A} : f^* \geq h} U(h) \leq u(x_m) < +\infty.$$

Similarly, $U(f^*) > -\infty$. As $u \circ f = u \circ f^* + u \circ f_1$, then $U(f)$ is finite as well. □

It remains to show that the expected utility function $U$ represents the preference $\succ$, that is, for all acts $f, g \in \mathcal{F}$,

$$U(f) \geq U(g) \Longleftrightarrow f \succ g.$$

If the range of the function $u$ has one or two distinct values, then representation (11) follows from Monotonicity and Lemma 6. Assume without loss in generality that the range of $u$ has at least three distinct values. Take $x_1, x_0, y_1 \in X$ such that $x_1 > x_0 > y_1$. Next, for all $i = 2, 3, \ldots$ take $x_i, y_i \in X$ such that

$$X = \bigcup_{x \in X : x \ni y_i} \{ z \in X : x \ni z \ni y_i \}$$

and $\ldots \ni x_3 \ni x_2 \ni x_1 \ni x_0 \ni y_1 \ni y_2 \ni y_3 \ni \ldots$. Indeed, if there exists $x \in X$ such that $x \ni x'$ for all $x' \in X$, then one can take $x_2 = x_3 = \ldots = x$. Otherwise, if for any $x \in X$ there exists $x' \in X$ such that $x' \ni x$, then one can take $x_2 \ni x_3 = \ldots$ such that $\lim_{n \rightarrow \infty} u(x_n) = \sup_{x \in X} u(x)$. The outcomes $y_2, y_3, \ldots$ can be defined similarly.

Consider several possible cases.

Case 1. $U(f^*) > U(g)$ and $g \succ f$. For all $i = 1, 2, \ldots$, let

$$A_i = \{ s \in S : f(s) > x_i \text{ or } y_i \ni f(s) \}, \quad f_i = x_1 A_i f, \quad C_i = \{ s \in S : g(s) > x_i \text{ or } y_i \ni g(s) \}, \quad g_i = x_1 C_i g.$$

As $A_i \ni \emptyset$, then by SMC there exists $j$ such that $U(f_j) > U(g)$ and $g_j \ni f_j$. Similarly, there exists $j$ such that $U(f_j) > U(g_j)$ and $g_j \ni f_j$. Both acts $f_j$ and $g_j$ are bounded because $x_i \ni f_j \ni g_j$ and $y_i \ni f_j \ni g_j$. By Lemma 7, there exist simple acts $h^*, h_1 \in \mathcal{A}$ such that $U(f_j) \geq U(h_1) > U(h^*) \geq U(g_j)$ and $h^* \ni g_j \ni f_j$. By Monotonicity, $h^* \ni h_1$, which contradicts Lemma 6.

Case 2. $U(f) > U(g)$ and $g \ni f$. Let $A_1 = \{ s \in S : f(s) < x_1 \}$ and $C_1 = \{ s \in S : g(s) > y_1 \}$. Let $B = \{ B \in \mathcal{A} : p(B) = 0 \}$. Suppose that $p(A_1) > 0$. As $B$ is union-closed, then by Lemma 5, there are events $A_2, A_3, \ldots \in \mathcal{A} \setminus B$ such that $A_i \ni \emptyset$. By SMC, there is $i$ such that $g_i \ni f_i$. As $p(A_1) > 0$ and $u(f(s)) \ni u(y_1)$ for all $s \in C_i$, then $U(f_i) \geq U(g) > U(y_1 C_i f_i)$. By Case 1, this reversal is impossible. Therefore, $p(A_1) = 0$. Suppose that $p(C_1) > 0$. By Lemma 5, there are events $C_2, C_3, \ldots \in \mathcal{A} \setminus B$ such that $C_i \ni \emptyset$. By SMC, there is $i$ such that $y_i C_i g \ni f_i$. As $p(C_1) > 0$ and $u(g(s)) \ni u(y_1)$ for all $s \in C_i$, then $U(f_i) \geq U(g) > U(y_1 C_i f_i)$. By Case 1, this reversal is impossible. Therefore, $p(C_1) = 0$. As $p(A_1) = p(C_1) = 0$, then $U(f) \geq U(x_1) > u(y_1) \geq U(g)$. By Case 1, this strict inequality cannot hold together with $g \ni f$. 

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Case 3: $U(g) > U(f)$ and $f \succneq g$. Let $A_1 = \{s \in S : f(s) < x_0\}$ and $C_1 = \{s \in S : g(s) > x_0\}$. Suppose that $p(A_1) > 0$. By Lemma 5, there are events $A_2, A_3, \ldots \in \mathcal{A} \setminus \mathcal{B}$ such that $A_i \nrightarrow \emptyset$. As $p \in \mathcal{P}_0$ is countably additive, then $\lim_{i \to \infty} p(A_i) = 0$. It follows from Billingsley (1995, Theorem 16.9) that there is $i$ such that

$$p(A_i)u(x_1) - \int_{A_i} u(f(s)) \, dp < U(g) - U(f).$$

Then $U(g) > U(x_1A_f) = U(f) + p(A_i)u(x_1) - \int_{A_i} u(f(s))$. As $p(A_i) > 0$, then $x_1A_i x_0 > x_0$. By STP, $x_1A_f > x_0A_f$. By Monotonicity, $x_0A_f \succ f$. Thus $x_1A_f \succ g$. By Case 1, this strict preference cannot hold together with $U(g) \succ U(x_1A_f)$. Thus $p(A_1) = 0$. Suppose that $p(C_1) > 0$. By Lemma 5, there are events $C_2, C_3, \ldots \in \mathcal{A} \setminus \mathcal{B}$ such that $C_i \nrightarrow \emptyset$. Then there is $i$ such that

$$p(C_i)u(y_1) - \int_{C_i} u(g(s)) \, dp > U(f) - U(g).$$

Then $U(y_1C_1g) = U(g) + p(C_i)u(y_1) - \int_{C_i} u(g(s)) > U(f)$. As $p(C_i) > 0$, then $y_1C_i x_0 > x_0$. By STP, $y_1C_i \succ x_0C_i$. By Monotonicity, $x_0C_i \succ g$. Thus $y_1C_i \nrightarrow g$. By Case 1, this strict preference cannot hold together with $U(y_1C_1g) \succ U(f)$. Thus $p(C_1) = 0$. As $p(A_1) = p(C_i) = 0$, then $U(f) \geq u(x_0) \geq U(g)$. This contradicts $U(g) > U(f)$.

By contradiction, equivalence (11) must hold for all acts $f, g \in \mathcal{F}$.

**A.1. Proofs of Corollaries 3 and 4**

Let $\mathcal{A} = \mathcal{I}$ and $\mathcal{F} = \mathcal{F}(\mathcal{I})$. Suppose that $\succ$ satisfies Axioms 1–5. I claim that $\succ$ satisfies Axioms 1–4 if $\mathcal{A}$ is replaced by $\alpha(\mathcal{A})$, which is the minimal algebra that contains $\mathcal{I}$. By definition, $\mathcal{F}(\alpha(\mathcal{A}))$ equals the set $\mathcal{F}(\alpha(\mathcal{I}))$ of all $\alpha(\mathcal{I})$-measurable simple acts. Thus $\succ$ is complete and transitive on $\mathcal{F}(\alpha(\mathcal{I}))$. Take any $A \in \alpha(\mathcal{I})$. Then $A = \cup_{i=1}^{\infty} \{a_i, b_i\}$ for some $0 \leq a_1 \leq b_1 \leq a_2 \leq b_2 \leq \ldots \leq a_n \leq b_n \leq +\infty$. By STP, for all acts $f, g, h, h' \in \mathcal{F}$ and all $i = 1, \ldots, n$,

$$f(a_i, b_i)h > g(a_i, b_i)h \Rightarrow f(a_i, b_i)h' \succ g(a_i, b_i)h'. $$

Thus by induction with respect to $n$, $f \mathcal{A} h \Rightarrow g \mathcal{A} h \Rightarrow f \mathcal{A} h' \succ g \mathcal{A} h'$

and hence, STP for all $A \in \alpha(\mathcal{A})$.

Take any $A, A' \in \alpha(\mathcal{A})$ and outcomes $x > y$ and $x' > y'$. Suppose that $xAy \succ xA'y$. By STP, $x(A \setminus A')y \succ x(A' \setminus A)y$. Note that $A \setminus A' = \cup_{i=1}^{\infty} \{a_i, b_i\}$ and $A \setminus A' = \cup_{i=1}^{\infty} \{a_i', b_i'\}$, where $0 \leq a_1 \leq b_1 \leq \ldots \leq a_n \leq b_n \leq +\infty$ and $0 \leq a_1' \leq b_1' \leq \ldots \leq a_m \leq b_m \leq +\infty$. I claim that $x'(A \setminus A')y' \succ x'(A' \setminus A)y'$. The proof is by induction with respect to $m + n$. If $m = n = 1$, then this claim follows from Comparative Probability for $\mathcal{A}$. Take arbitrary $m$ and $n$. Suppose that $x[a_n, b_n]y \succ x[a_m, b_m]y$. Let $c_n = \sup_{a \in [a_n, b_n]} x[a, c]y \sim x[a_n, b_n]y$. Then by SMC, $x[a_n, c]y \sim x[a_n, b_n]y$. By Comparative Probability, $x[a_1, c]y \sim x[a_n, b_n]y$. Let $B = (\cup_{i=1}^{m-1} \{a_i, b_i\}) \cup \{a_n, c\}$ and $B' = (\cup_{i=1}^{m-1} \{a_i', b_i'\})$. By STP, $x'(A \setminus A')y' \succ x'(A' \setminus A)y'$ implies $xBy \succ x'B'y$. By the inductive assumption, $xBy \succ x'B'y$. Thus $x(A' \setminus A'y') \succ x'(A' \setminus A)y'$. This Comparative Probability holds for all $A, A' \in \alpha(\mathcal{I})$.

Finally, recall the construction of events $A_1, A_2, \ldots$ such that $A_1 = S$ and $A_i \nrightarrow \emptyset$ in the proof of Lemma 6. When $S = [0, +\infty)$, then all of these events can be taken intervals of the form $A_i = [a_i, b_i]$ for some $a_i \leq b_i$. Thus Lemma 6 holds and $\succ$ has the required expected utility representation (5) for some $u \in \mathcal{U}$ and $p \in \mathcal{P}_0$.

To show Corollary 4, suppose that $\succ$ satisfies Stationarity. Fix outcomes $x > y$. Then $p(a, b) > 0$ for all $0 \leq a < b \leq +\infty$. Indeed, if $p(a, b) = 0$, then $x[a, b]y \sim y$. By Stationarity for all $t \geq -a, x[a+t, b+t]y \sim y$ and hence, $p(a+t, b+t) = 0$. By countable additivity, $p(0, +\infty) = 0$, which is a contradiction.

For all $s \in [0, +\infty)$, let

$$F(s) = p([s, +\infty)).$$

Then $F$ is a strictly decreasing function such that $F(0) = 1$ and $\lim_{s \to \infty} F(s) = 0$. Moreover, $F$ is continuous because $p \in \mathcal{P}_0$. For all $t \in [0, +\infty)$, let

$$F(t) = \frac{F(t + s)}{F(t)}.$$ 

By construction, $F_t$ is also a strictly decreasing continuous function such that $F_t(0) = 1$ and $\lim_{s \to \infty} F_t(s) = 0$. I claim that $F_t(s) = F(s)$ for all $s \in [0, +\infty)$. Suppose that $F(s) = (1/2)$. Then $x[0, s]y \sim x[s, +\infty)y$ implies by Stationarity that $x[t, t+s]y \sim x[t+s, +\infty)y$. Thus $p(t, t+s) = p(t+s, +\infty)$ and hence, $F_t(s) = (1/2)$. Suppose next that $F(s) = (2k-1)/2^n$ where $k = 1, \ldots, 2^n - 1$. Let $s' \succ$ such that $F(s') = (k-1)/2^n$ and $F(s') = (k-1)/2^n$. (If $k = 1$, then take $s' = +\infty$.) Assume that $F(s') = F(s')$. Thus $F(s') = F(s')$. Then $x(s', s) = x(s', s')y$ implies by Stationarity that $x[t+s, t+s']y \sim x[t+s', t+s']y$. It follows that $F(t+s) - F(s) = F(s) - F(s)$ and hence, $F_t(s) = (2k-1)/2^n$. By induction with respect to $n$, $F_t(s) = F(s)$ for all $s$ such that $F(s) = (2k-1)/2^n$ for some $n$ and $k = 1, \ldots, 2^n - 1$. As the binary rational numbers are dense in $[0, 1]$ and both functions $F$ and $F_t$ are continuous and strictly monotonic, then the inverse functions $F^{-1}$ and $F_t^{-1}$ are also continuous, and $F_t(s) = F(s)$ for all $s \in [0, +\infty)$.
By Cauchy’s equation (see Billingsley, 1995, pp. 540–541), \( f(s) = e^{-\gamma s} \) for some \( \gamma > 0 \). The uniqueness of \( \gamma \) follows from the uniqueness of \( p \) and \( F \).

### A.2. Convex-rangedness, fine-rangedness, and the nullity of singletons

The following corollary clarifies the connection between convex-rangedness and the inclusion \( p \in \mathcal{P}_0 \) in Theorem 1.

**Lemma 8.** If \( A \) is a countably separated algebra and \( p \in \mathcal{P} \), then the following statements are equivalent:

(i) \( p(\{s\}) = 0 \) for all \( s \in S \),

(ii) \( p \) is convex ranged: for all \( B \in \mathcal{E}(A) \),

\[
\{ p(A) : A \in \sigma(A), A \subset B \} = [0, p(B)],
\]

(iii) when restricted to \( A \), \( p \) is finely ranged: for any \( \varepsilon > 0 \), the state space \( S \) can be partitioned into finitely many events \( A_1, \ldots, A_m \in A \) such that \( \| p(A_i) \| < \varepsilon \) for all \( i \),

(iv) \( p \) is non-atomic: for any \( B \in \sigma(A) \) such that \( p(B) > 0 \), there is \( A \in \sigma(A) \) such that \( A \subset B \) and \( 0 < p(A) < p(B) \).

In particular, this result strengthens Aliprantis and Border’s (1999, Lemma 10.17) that establishes the equivalence of non-atomicity and the inclusion \( p \in \mathcal{P}_0 \) for Borel measures in a second countable Hausdorff spaces. Note that \( p \) need not be convex ranged on the primitive algebra \( A \) (see examples in Kopylov, 2007).

Let \( A \) be a countably separated algebra, and \( p \in \mathcal{P}_0 \). Let \( X = \{ x, y \} \) and define a preference \( \succeq \) over all \( \mathcal{E}(A) \) measurable acts by

\[
xAy \succeq xBy \iff p(A) \geq p(B).
\]

Then \( \succeq \) has the utility representation (5) and hence, satisfies Axioms 1–5. The proof of Lemma 6 shows that these axioms imply P6. Therefore, \( p \) is convex ranged by Savage’s theorem. The implication (ii) \( \Rightarrow \) (iii) follows from Kopylov’s (2007, Theorem 3.1). The implications (ii) \( \Rightarrow \) (iv) \( \Rightarrow \) (i), and (iii) \( \Rightarrow \) (i) are obvious.

**References**


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6 This terminology is used by Kopylov (2007). Rao and Rao (1983) call such measures strongly continuous.