INTRODUCTION

Historically, the term probability has been associated with a number of different concepts. The standard mathematical definition due to Kolmogorov (1933/1950) identifies probability with a countably additive measure on a \( \sigma \)-algebra.\(^1\) This definition underlies a powerful theory that includes the law of large numbers and the central limit theorem. Yet in the applications of this theory, probability has different extramathematical interpretations.

Since its inception by Bernoulli (1713), de Moivre (1756), and Laplace (1814/1917), the calculus of probability has been applied to—or even designed for—repetitive experiments with gambling apparatus, such as coins, cards, dice, roulette wheels, and so on. Each of these devices has a natural symmetry that is commonly known a priori. This symmetry determines an exhaustive collection of equally possible states and motivates the classical interpretation of probability as “the fraction whose numerator is the number of favorable cases and the denominator is the number of all the cases possible” (Laplace, 1917, pp. 6–7).

More broadly, probability theory has been used to analyze repetitive observations of various physical, biological, and social systems for which one cannot find any compelling symmetry a priori. Motivated by such applications, Ellis (1863) and Venn (1866/1962) proposed to identify probability of any observed attribute with the limit of the relative frequency of this attribute in a long, ideally infinite, series of trials. Venn (1866/1962) writes,

> The run must be supposed to be very long indeed, in fact never to stop. As we keep on taking more terms of the series, we shall find this proportion [i.e. the relative frequency] still fluctuating a little, but its fluctuations will grow less. The proportion, in fact, will gradually approach toward some numerical value, . . . its limit. (p. 164)

Note that the classical and the frequentist interpretations of probability complement rather than contradict each other. For example, the statistics of Bose-Einstein and Fermi-Dirac are both

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\(^1\)A \( \sigma \)-algebra is a class of events that is closed under taking complements and countable unions. See Billingsley (1995) for an excellent modern introduction into probability theory.
derived from a priori symmetry considerations but then matched with real physical particles (photons and electrons, respectively) via observed frequencies.

Another application for mathematical probability has been to quantify human beliefs. In this area, neither the classical nor the frequentist interpretation of probability seems satisfactory. Indeed, beliefs need not be based on a priori symmetry or observed relative frequencies. For example, I think that in 20 years (1) the U.S. income tax rates are more likely to go up than down and (2) the global climate is more likely to get warmer than colder. However, I derive these beliefs from articles in Wall Street Journal and National Geographic and from superficial and vague perception of recent budget deficits and unusual weather conditions. Is it meaningful to describe my (or anybody else’s) beliefs in terms of mathematical probability?

Keynes (1921) and Carnap (1950) propose to answer this question on the premise that beliefs are derived from the available evidence via universally acceptable principles. In their interpretation, probability is an objective relation between propositions—a relation that by logical necessity describes all reasonable beliefs. This logical approach suffers from two serious practical limitations. First, the available evidence can be informal, vague, and voluminous (think about a transcript of a court trial!). Second, there are hardly any undisputable principles that translate such evidence into beliefs. Instead, most beliefs appear to come from an impenetrable mix of instincts, intuition, experience, analogy—a mix that defies a satisfactory logical analysis. As a result, beliefs are often idiosyncratic, even if their owners share the same meticulously controlled evidence (think about a jury in a court trial!). In the words of Savage (1972),

there is no fundamental objection to the possibility of constructing a necessary [i.e. logical] view, but it is my impression that the possibility has not yet been realized, and though unable to verbalize the reasons, I conjecture that the possibility is not real. (p. 61)

To break through this impasse, Ramsey (1931) proposes to associate beliefs with decisions rather than with the underlying evidence: “...the degree of a belief is a causal property of it, which we can express vaguely as the extent to which we are prepared to act on it” (p. 170). Indeed, beliefs do affect choices that people actually make, or plan to make, in suitable circumstances. For example, subjective judgments about the future income tax rates and global warming effects can influence how much a person chooses to save and what car he chooses to drive. In general, it seems that almost any human belief should manifest itself in some material choice behavior, which, at least in principle, can be observed by a modeler.

Building on the seminal contributions of Ramsey (1931), de Finetti (1937), and Savage (1972), the theory of subjective probability seeks to describe choice under uncertainty in terms of probabilistic beliefs. In particular, this theory seeks a probability measure \( p \) to represent a betting preference \( A \succeq_0 B \) so that for all events \( A \) and \( B \),

\[
A \succeq_0 B \iff p(A) \geq p(B). \tag{3.1}
\]

Here the decision maker exhibits the preference \( A \succeq_0 B \) if, ceteris paribus, he or she chooses to bet on the event \( A \) rather than on the event \( B \). Intuitively, this choice manifests his or her belief that \( A \) is at least as likely as \( B \). Thus, the sought-after probability measure \( p \) should reflect exclusively personal beliefs rather than any objective characteristics of the physical world.

Representation (3.1) interprets subjective probability in terms of choice among bets that have dichotomous (i.e., the good and the bad) outcomes. However, uncertain prospects that people face in everyday life—as well as in models of game theory, finance, economics, and other sciences—usually have three or more possible outcomes. Does it make sense to use subjective probability to evaluate such prospects? What sort of evaluation is reasonable? These questions are addressed by Savage’s theory of subjective expected utility, which is distinguished by Kreps (1988) as “the crowning glory of choice theory.” This theory portrays a person who (1) assigns subjective probabilities \( p(\cdot) \) to relevant events, (2) assigns a utility index \( u(\cdot) \) to relevant outcomes, and (3) evaluates every uncertain prospect \( f \) via the expected utility

\[
U(f) = \sum_{\text{outcomes } x} u(x) \cdot p(\{f \text{ yields } x\}). \tag{3.2}
\]
Subjective and Other Probabilities

In contrast with the classic treatments of expected utility by Bernoulli (1713) and by von Neumann and Morgenstern (1944), the probability measure \( p \) is not given exogenously but derived from personal choice. In particular, decision makers may differ not only in their tastes over possible outcomes but also in their assessment of probabilities.

An important special case of subjective expected utility obtains when the outcomes \( x \) are monetary and the decision maker is risk neutral with the utility index \( u(x) = x \). Then, the decision maker’s subjective probability \( p(A) \) equals the price at which he or she is willing to buy or sell a ticket that pays a unit of money if \( A \) occurs and nothing otherwise. Following de Finetti, some authors have used this equality as the basic definition of subjective probability. In this way, one can derive probabilities directly from pricing decisions that are well understood by most people. Moreover, if probabilities are identified with prices, then it is straightforward to argue that they must be additive because the decision maker must avoid Dutch books, that is, losing money for sure. Unfortunately, the assumptions that outcomes are monetary and people are risk neutral are too restrictive for many applications. Thus, to construct subjective probabilities \( p(\cdot) \) in the general representations (3.1) and (3.2), one needs devices more subtle than monetary prices.

The mathematical definition alone is a rather minimalist constraint for subjective probability. It does not exclude unrealistic beliefs that do not correlate with experience. Thus, it may be desirable to derive additional structure for subjective probability in applications. For example, it is intuitive that personal beliefs should converge to empirical frequencies in the long run. Formally, this property can be obtained by de Finetti’s theorem or by the market selection argument.

The general theory of subjective probability is normative: it seeks to explain why an idealized rational person should use a probabilistic belief to make hypothetical decisions under uncertainty. Thus, it is neither very surprising nor completely disappointing that the findings of this theory have little empirical support. One problem is that abstract decision frameworks where subjective probability is derived do not fit empirical settings very well. For example, a person who moves to work in another city faces an uncertain prospect that is not easily represented as a Savage-style function that maps states of the world into outcomes. Even if the person himself or herself were able to write such a representation, this task may be still problematic for the modeler. On the other hand, the construction of subjective probability requires so many hypothetical decision problems that one could not possibly observe a personal response in each of them. In fact, a large portion of these problems are so unrealistic and complex that they cannot be a part of any experiment.

Moreover, there is a lot of direct empirical evidence that violates the representations (3.1) and (3.2). Such violations have been reported by Allais (1953), Ellsberg (1961), Kahneman and Tversky (1979), and many others. Do these departures refute the theory of subjective probability? Put differently, is it possible to modify this theory to accommodate the observed patterns of choice behavior? The discussion of these questions will conclude this chapter.

**SUBJECTIVE PROBABILITY AND BETTING PREFERENCES**

Following Savage (1972), consider a set \( S \) of states of the world and a person who knows that there exists a unique true state in \( S \), but does not know the identity of the true state with certainty. In other words, assume that the person (also known as the subject or the decision maker) views the state space \( S \) as an exhaustive list of mutually exclusive descriptions of the world.\(^2\)

Uncertainty in this framework is atemporal and may relate to the past, present, and future of the universe, as well as to the abstract realm of human cognition. For example, a state \( s \in S \) may specify

- the number that will be hit by the next spin of a roulette wheel,

the next winner of the FIFA World Cup,
• the spot where the next hurricane will strike the American shore,
• the decimal expansion of the number π, and
• whether Abraham Lincoln was the 16th president of the USA.

To incorporate all of these different aspects of the world into one state space, it is natural to use a product structure and take \( S = S_1 \times S_2 \times S_3 \times S_4 \times S_5 \) with

- \( S_1 = \{0,0,1,2,\ldots,36\} \) (allowing for a double zero),
- \( S_2 \) the set of all FIFA members,
- \( S_3 \) the set of all geodesic points along the American shore,
- \( S_4 = [3,4] \),
- \( S_5 = \{\text{yes, no}\} \).

Note that the source of uncertainty can be purely cognitive: for example, the fact that Lincoln was the 16th president is known to many people but may still appear uncertain to a participant in a quiz or a student at a closed-book exam.

Assume that \( S \) is a separable metric space. In particular, it can be a finite set, a subset of a Euclidean space, or a finite or countable product of such sets because all the corresponding topologies—discrete, Euclidean, and product—are separable and metrizable.

Let \( \Sigma \) be the Borel \( \sigma \)-algebra on \( S \), and call its elements \( A \in \Sigma \) events. Say that an event \( A \in \Sigma \) occurs if it contains the true state of the world. Note that \( S \) is the universal event that occurs with certainty, and \( \emptyset \) is the vacuous event that never occurs.

Adopt Kolmogorov’s definition and say that a real function \( p : \Sigma \to \mathbb{R} \) is a probability measure if it obeys the following properties for all \( A, B, A_i \in \Sigma \):

(K1) \( p(A) \geq 0 \),
(K2) \( p(S) = 1 \),
(K3) if \( A \cap B = \emptyset \), then \( p(A \cup B) = p(A) + p(B) \),
(K4) if \( A_i \uparrow A \), then \( p(A) = \lim_{i \to \infty} p(A_i) \),

where the convergence \( A_i \uparrow A \) means that \( A_1 \supseteq A_2 \supseteq \cdots \) and \( A = \cap_{i=1}^{\infty} A_i \).

Let a binary relation \( \succeq_0 \) on \( \Sigma \) be the subject’s betting preference. Formally, for any event \( A \), let \( t_A \) be a ticket that yields a unit of money—say \$1—if \( A \) occurs, and nothing otherwise. Write \( A \succeq_0 B \) if the subject accepts the ticket \( t_A \) when \( t_B \) is a feasible alternative. Note that \( A \succeq_0 B \) and \( B \succeq_0 A \) may hold together, which means that the subject is indifferent between the tickets \( t_A \) and \( t_B \). Let the indifference \( \sim_0 \) and the strict preference \( \succ_0 \) denote the symmetric and asymmetric parts of \( \succeq_0 \), respectively.

Intuitively, a preference \( A \succeq_0 B \) to bet on an event \( A \) rather than on \( B \) manifests the decision maker’s implicit belief that the event \( A \) is at least as likely as the event \( B \). Therefore, to quantify such beliefs by probabilities, one may seek a probability measure \( p \) such that for all events \( A \) and \( B \),

\[
A \succeq_0 B \iff p(A) \geq p(B). \tag{3.3}
\]

In other words, the sought-after probability \( p \) is a utility representation for the personal ranking of bets. When does such \( p \) exist? When is it unique? These questions have been addressed by de Finetti (1937), Koopman (1940), Savage (1972), Scott (1964), Luce (1967), Villegas (1964), Wakker (1981), and others.

Any binary relation \( \succeq_0 \) that complies with representation (3.3) must satisfy the following list of conditions for all events \( A, A_i, B, C \in \Sigma \). (This list is a compilation from de Finetti, 1937 and Villegas, 1964.)

(L1) Completeness: \( A \succeq_0 B \), or \( B \succeq_0 A \) (or both).
(L2) Transitivity: If \( A \succeq_0 B \succeq_0 C \), then \( A \succeq_0 C \).
(L3) Monotonicity: \( A \succeq_0 \emptyset \).
(L4) Nondegeneracy: \( S \supset _0 \emptyset \).
(L5) Additivity: If \( A \cap C = B \cap C = \emptyset \), then \( A \succeq_0 B \) is equivalent to \( A \cup C \succeq_0 B \cup C \).
(L6) Continuity: If \( A_i \downarrow A \) and \( A_i \succeq_0 B \), then \( A \succeq_0 B \).
Here completeness and transitivity are well-known general postulates of rational choice, while the conditions of monotonicity, nondegeneracy, additivity, and continuity are more specific for the choice among bets. In particular, additivity asserts that the preference to bet on the events $A$ and $B$ should be unaffected by whether a monetary prize is paid contingent on an event $C$ that is disjoint from both $A$ and $B$. Continuity requires roughly that bets on events $A$ and $A_i$ in a converging sequence $A_i \uparrow A$ are almost imperceptible for sufficiently large $i$.

Despite having a direct analogy with Kolmogorov’s definition, the list (L1) to (L6) does not guarantee that representation (3.3) exists. Kraft, Pratt, and Seidenberg (1959) formulate a counterexample on a five-element state space $S = \{1, 2, 3, 4, 5\}$. They show that conditions (L1) to (L6) are consistent with the rankings

\[
A_1 = \{4\} \succ \{1, 3\} = B_1,
A_2 = \{2, 3\} \succ \{1, 4\} = B_2,
A_3 = \{1, 5\} \succ \{3, 4\} = B_3,
A_4 = \{1, 3, 4\} \succ \{2, 5\} = B_4.
\]

By definition, any probability measure $p$ satisfies

\[
\sum_{i=1}^{4} p(A_i) = 2p(\{1\}) + p(\{2\}) + 2p(\{3\}) + 2p(\{4\}) + p(\{5\}) = \sum_{i=1}^{4} p(B_i),
\]

and hence, $p(B_i) \geq p(A_i)$ for some $i$. Thus $p$ does not represent $\succeq$.

Moreover, even if representation (3.3) does exist, it need not be unique. For example, the ranking $S = \{a, b\} \succ \{a\} \succ \{b\} \succ \emptyset$ can be quantified by any probability measure $p$ such that

\[
1 > p(\{a\}) > \frac{1}{2}.
\]

To guarantee existence and uniqueness of representation 3.3, impose an extra condition on the preference $\succeq_0$.

(L7) **Nondiscreteness:** For every $s \in S$, $\{s\} \succ_0 \emptyset$.

This condition asserts that the subject should view any single state $s \in S$ as practically impossible, and the corresponding ticket $t_{(s)}$ as worthless. For example, nondiscreteness seems intuitive if every state $s \in S$ specifies an infinite sequence of coin flips or similar independent random experiments. Unfortunately, nondiscreteness must be violated if $S$ is finite.

Now the sought-after representation (3.3) can be derived from the results of Savage (1972) and Villegas (1964).

**Theorem 3.1.** $\succeq_0$ satisfies (L1) to (L7) if and only if $\succeq_0$ can be represented by a probability measure $p$ such that $p(\{s\}) = 0$ for all $s \in S$. This representation is unique.

One can view this theorem as a foundation for the use of subjective probabilities in decision making. Here, probabilities are (i) consistent with Kolmogorov’s definition and (ii) purely subjective because they are derived exclusively from personal choice behavior rather than from any exogenous numerical representation of uncertainty. In particular, two people may both comply with the conditions of Theorem 3.1 but differ in their assignments of probabilities.

The construction of the probabilities $p(\cdot)$ in Theorem 3.1 can be done via an elegant formula, which reveals another connection between betting preferences and the underlying beliefs. Call any partition of the universal event $S$ into disjoint events $S_1, \ldots, S_m$ a **grand partition**. Say that a grand partition is **finer** than an event $A$ if $A \succ_0 S_i$ for all $i$, that is, if all events $S_i$ are subjectively less likely than $A$. Among all grand partitions finer than $A$, take one with a minimal number of elements. Let $v(A)$ be this minimal number; let $v(A) = +\infty$ if there is no partition finer than $A$. Then the unique probabilities $p(A)$ for all events $A$ are given by the formula

\[
p(A) = \sup \left\{ \sum_{i=1}^{n} \frac{1}{v(A_i)} \right\},
\]

where the supremum is taken across all partitions of $A$ into disjoint events $A_1, \ldots, A_n$. (See Kopylov, 2007 for details.)

Note that the original results of Savage and de Finetti derive a **finitely additive** subjective probability measure, which satisfies (K1) to (K3) but may violate (K4). The restriction to countable additivity in Theorem 3.1 has two advantages. First, it permits the use of subjective probability in the law of large numbers, the central limit theorem,
and other standard results that require countable additivity. Second, the “technical” conditions L6 and L7 in Theorem 3.1 appear more transparent than their counterparts, such as Savage’s fineness and tightness, in the finitely additive model.

Unfortunately, Theorem 3.1 fails when $S$ is finite and hence, $\geq_0$ violates nondiscreteness. To accommodate this case, Scott (1964) strengthens additivity. He postulates that if every state $s \in S$ belongs to as many events in a list $A_1, \ldots, A_n$ as in another list $B_1, \ldots, B_m$, and if $A_i \geq_0 B_i$ for all $i < n$, then $B_n \geq_0 A_n$. Obviously, this postulate rules out the counterexample of Kraft et al. (1959). Scott shows that this stronger form of additivity, together with (L1) to (L4), is necessary and sufficient for the preference $\geq_0$ to have representation (3.3). As mentioned above, this representation need not be unique.

Alternatively, one can follow Luce (1967) and obtain subjective probabilities via the theory of extensive measurement. This approach applies to both finite and infinite settings, but in the finite case it requires essentially that all states are equally likely.

Instead of confronting a person with monetary bets, one can simply ask him or her which of two events $A$ or $B$ he or she views as more probable. By this kind of interrogation, one can also derive a subjective ranking $\geq_0$ of likelihoods of events and then seek a probability measure $p$ to represent this ranking. This intuition-oriented approach is adopted by Koopman (1940) and in part, by de Finetti (1937).

Of course, the formal statements and proofs of Theorem 3.1 and similar representation results are unaffected by the extramathematical interpretation that one adopts for the ranking $\geq_0$. However, the intuition- and decision-oriented interpretations do have some practical distinctions. First, most people find material decisions more important and more suitable for scientific analysis than intuitive judgments. This point is emphasized by Savage (1972):

Many doubt that the concept “more probable to me than” is an intuitive one, open to no ambiguity and yet admitting further analysis. Even if the concept were so completely intuitive, which might justify direct interrogation as a subject worthy of some psychological study, what could such interrogation have to do with the behavior of a person in the face of uncertainty, except of course for his verbal behavior under interrogation? If the state of mind in question is not capable of manifesting itself in some sort of extraverbal behavior, then it is extraneous to our main interest. If, on the other hand, it does manifest itself through more material behavior, that should, in principle, imply the possibility of testing whether a person holds one event more probable than another, by some behavior expressing, and giving meaning to, his judgement. (p. 27)

Moreover, in some settings betting preferences may differ persistently from the intuitive perception of probabilities. For example, a person may believe that she is more likely to retire rich rather than poor, but still prefer to bet on the latter event. In this case, the personal value of a monetary payoff clearly depends on the event where this payoff is obtained. See Karni (1993) for a model of subjective probability with state-dependent preferences.

**SUBJECTIVE EXPECTED UTILITY**

Subjective probability in Theorem 3.1 represents personal choices among bets—uncertain prospects that have only two possible outcomes, such as $\$1$ and $\$0$. However, people often face more complex decision problems where more than just two outcomes appear possible. For example, potential consequences of economic or financial decisions may include a whole range of monetary payments or consumption bundles, which can be written as real numbers or vectors, respectively. Any game between a player with $m$ strategies and a player with $n$ strategies has up to $mn$ distinct outcomes, which may have a purely verbal description (think about various trips to ballet and boxing in the battle of the sexes). Is it still meaningful to apply subjective probabilities in these more complex settings?

To address this question formally, let $X$ be the set of all outcomes (payoffs, prizes) that the subject may obtain after his or her decisions are made, and the true state of the world is revealed. Assume that each of these outcomes can be potentially experienced in any state of the world.4

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4Nau (2001) reviews the critique of Savage’s concept of outcomes. Skiadas (1997) formulates a model of subjective probability that does not take outcomes as given.
Interpret any function \( f : S \to X \) as an uncertain prospect that yields the outcome \( f(s) \) when \( s \) is the true state of the world; call this function an act. Formally, require that each act \( f \) is a Borel function that has a finite range in \( X \). Given any event \( A \) and acts \( f, g \), let \( fA g \) be a composite act that yields \( f(s) \) if \( s \in A \) and \( g(s) \) if \( s \not\in A \).

For example, a constant act \( x \) represents an action that yields the same payoff \( x \) in any state of the world, and a binary act \( t_A = \$1 \) if event \( A \) occurs and \$0 otherwise. To illustrate the use of more complex acts, Savage portrays an omelette maker who has broken five eggs into a bowl and needs to decide what to do with the remaining sixth one. This subject contemplates an event \( E = \{ \text{the sixth egg is rotten} \} \) in a suitable state space, and then identifies three different actions “to break the egg in the bowl,” “to throw the egg away,” and “to break the egg in a separate saucer for inspection” with the following acts \( f, g, \) and \( h \):

\[
\begin{array}{ccc}
  f & g & h \\
  s \in E & \text{No-egg} & \text{Five-egg} & \text{Five-egg} \\
  \text{omelette} & \text{omelette} & \text{omelette} \\
  s \not\in E & \text{Six-egg} & \text{Five-egg} & \text{Six-egg} \\
  \text{omelette} & \text{omelette} & \text{omelette}
\end{array}
\]

Here, the acts \( f, g, h \) map states of the world into the set of outcomes

\[ X = \{ \text{no omelette, five-egg omelette, six-egg omelette} \} \]

Note that even in this simplistic world, some acts have little practical meaning. For example, consider \( f' \) that yields a six-egg omelette if \( E \) occurs and no omelette otherwise. While it is not hard to imagine \( f' \) as a hypothetical uncertain prospect, it is doubtful that such a strange bet has ever been proposed to any omelette maker.

Let a binary relation \( \succeq \) describe the personal preference over acts. More precisely, the comparison \( f \succeq g \) means that the person accepts the act \( f \) when \( g \) is a feasible alternative. Write the indifference and strict preference as \( \sim \) and \( \succ \), respectively.

There are two broad venues for studying preferences over acts. First, one may collect empirical data about choice behavior from surveys or laboratory experiments and then design models to fit this data well. Alternatively, one may search for normative principles that reasonable people—including the modeler—would want to obey both in real and in hypothetical decision problems. In Savage’s words, the primary goal of such conditions is “to police . . . decisions for consistency and, where possible, to make complicated decisions depend on simpler ones” (Savage, 1972, p. 20).

From the normative standpoint, the following conditions on the preference \( \succeq \) may be reasonable for all events \( A, B, A_i \), outcomes \( x, x', z, z' \), and acts \( f, g, h, h' \):

(P1) Order: \( \succeq \) is complete and transitive.

(P2) Sure-thing principle: If \( fAh \succeq gAh \), then \( fAh' \succeq gAh' \).

(P3) Monotonicity: If \( x \succeq x' \), then \( xAh \succeq xA h' \).

(P4) Comparative probability: If \( x \succ x', z \succ z' \), and \( xA x' \succeq xB x' \), then \( zA z' \succeq zB z' \).

(P5) Nondegeneracy: \( \succeq \) is not empty.

(P6) Nondiscreteness: \( x[s]f \sim f \).

(P7) Continuity: If \( A_i \uparrow A \) and \( h \succeq xA_i f \succeq g \), then \( h \succeq xA f \succeq g \).

Sure-thing principle asserts that the ranking of acts \( f \) and \( g \) that are conditioned on an event \( A \) is independent of the outcomes that are obtained if \( A \) does not occur. Monotonicity requires roughly that the ranking of outcomes in \( X \) is invariant of the event where these outcomes are obtained. Comparative probability states that the preference to bet on an event \( A \) rather than on \( B \) should be unaffected by the stakes that are involved in these bets. The other axioms in the above list are similar to their counterparts for the betting preferences.

While axioms P1 to P7 are compelling in some settings, they may be problematic in others. First, if events have a direct effect on the value of outcomes, then monotonicity may fail: for example, a person who prefers “six-egg omelette” with certainty to “no omelette” with certainty may reverse his or her preference if these outcomes are conditioned on the event “his or her cholesterol level is high”. Second, if acts affect the relative likelihoods of events (i.e., there is a moral hazard), then comparative probability may be violated: for example, a student may prefer to bet...
a million dollars on his passing an exam rather than on a coin flip, but reverse his or her preference if the monetary stake is just $1. Third, if the state space is finite, then nondiscreteness does not hold. Last but not the least, the axioms of order and sure-thing principle can be problematic for several other reasons as well, which we discuss later.

The following theorem is another nontrivial corollary to Savage’s results (see also Fishburn, 1970).

**Theorem 3.2.** \( \succeq \) satisfies conditions (P1) to (P7) if and only if \( \succeq \) can be represented by expected utility

\[
U(f) = \int_S u(f(s)) \, dp = \sum_{x \in X} u(x) \cdot p(\{s : f(s) = x\}),
\]

(3.5)

where \( u \) is a nonconstant utility index on \( X \), and \( p \) is a probability measure such that \( p(\{s\}) = 0 \) for all \( s \in S \). This representation is unique up to a positive linear transformation of the index \( u \).

The decision maker portrayed by (3.5) assigns probabilities \( p(\cdot) \) to all events \( A \), attaches utilities \( u(\cdot) \) to all outcomes \( x \), and then ranks all uncertain prospects \( f \) via expected utility. Thus Theorem 3.2 provides foundations for the use of subjective probability in a wide class of decision problems, not necessarily restricted to the ranking of bets. Moreover, it advocates the use of subjective probability as a component of the well-known expected utility functional form.

The expected utility criterion was originally proposed by Bernoulli (1713) as an ad hoc solution to the St. Petersburg paradox, and was characterized axiomatically by von Neumann and Morgenstern (1944). These authors assume that probabilities are objective and exogenous to decision making. Accordingly, one can view Theorem 3.2 as an extension of the expected utility theory to subjective probabilities. (In fact, Theorem 3.2 invokes von Neumann and Morgenstern’s result to obtain the utility index after deriving the probability measure \( p \) from preference.) Such an extension provides a useful flexibility in economic applications. For example, many people buy insurance even though they expect that by doing so, they will lose money on average. This phenomenon can be explained by a difference in attitudes toward risk: individuals are usually risk averse and hence, have a concave index \( u_1 \), while insurance companies are risk neutral (or almost so) and have a linear index \( u_2 \). On the other hand, even risk averse people with certain endowments may choose to bet against each other. Such betting can be explained within the expected utility paradigm but only if there is a difference in subjective beliefs.

The expected utility representation (3.5) allows different interpretations for the subjective probabilities \( p(\cdot) \). First, one can define a betting preference \( \succeq_0 \) by

\[
A \succeq_0 B \iff xAx' \succeq xBx' \quad \text{for all } x \succ x',
\]

and then check that \( \succeq_0 \) complies with conditions (L1) to (L7). Obviously, the measure \( p \) represents \( \succeq_0 \), and hence, can be computed via formula (3.4). This approach is used in the formal proof of Theorem 3.2.

Alternatively, a probability \( p(A) \) can be interpreted as a rate at which the decision maker evaluates the outcomes that are obtained contingent on the event \( A \). This interpretation appears in the famous essay of Bayes (1763): “The probability of any event is the ratio between the value at which an expectation depending on the happening of the event ought to be computed, and the value of the thing expected upon its happening.”

To make this interpretation formal, one needs to determine the utility index \( u \) prior to computing subjective probabilities. The easiest way to do so is to assume that outcomes are monetary and the utility over money is linear \( u(x) = x \). In other words, the person is assumed to be risk neutral, that is, indifferent between taking a monetary gamble and getting the mathematical expectation of this gamble for sure. Then the subjective probability \( p(A) \) equals the price at which he or she is willing to buy or sell the ticket \( t_A \) that yields a unit of money if \( A \) occurs and nothing otherwise. Indeed, it is easy to check that either of these transactions leaves his or her subjective expected utility unchanged. Thus under the assumption of risk neutrality, subjective probabilities \( p(\cdot) \) can be computed as prices. This method is clearly more practical than formula

\footnote{Pratt (1964) formulates the standard measurements of risk aversion in the expected utility model.}
(3.4): throughout the history of civilization, people have set monetary odds to express their beliefs in a credible, transparent, and numerically precise way.

Moreover, the identity between probabilities and prices permits a straightforward argument for additivity. This argument, due to Ramsey and de Finetti, asserts that a rational person should not accept a Dutch book, that is, a portfolio of bets that produces a sure loss in any state of the world. To avoid Dutch books, a person must set additive prices for tickets like $t_A$. Indeed, if $p(A) + p(B) < p(A \cup B)$ for some events $A$ and $B$, then the person will lose money for sure after selling $t_A$ and $t_B$ and buying $t_{A \cup B}$. Similarly, if $p(A) + p(B) > p(A \cup B)$, a person will end up with a Dutch book after buying $t_A$ and $t_B$ and selling $t_{A \cup B}$.

Note that pricing the ticket $t_A$ is not the only way for a risk neutral person to manifest his or her subjective probability $p(A)$. For example, consider a quadratic scoring rule, which Brier (1950) proposed for evaluating the accuracy of probabilistic weather forecasts. According to this procedure, the decision maker is asked to choose a number $\pi$ and then he or she is paid $-(\pi - 1)^2$ if $A$ occurs and $-\pi^2$ otherwise. To maximize his or her expected utility,

$$p(A) [- (\pi - 1)^2] + (1 - p(A)) [-\pi^2],$$

the person should choose the number $\pi$ that satisfies the first-order condition

$$2p(A)(\pi - 1) + (1 - p(A))\pi = 0,$$

that is, $\pi = p(A)$. It follows that this procedure provides incentives for the risk-neutral person to reveal his or her subjective probabilities truthfully.

Unfortunately, the assumptions that outcomes are monetary and people are risk neutral are too restrictive for many natural applications of subjective expected utility. For example, outcomes in game theory may have a purely verbal description; financial decisions, such as buying insurance or lottery tickets, are often incompatible with risk neutrality. Yet it can still make sense to derive any subjective probability $p(A)$ as a suitable rate of substitution. To do so, Anscombe and Aumann (1963) assume that outcomes of acts are objective probability distributions, called lotteries, ranked by von Neumann and Morgenstern’s expected utility measured in utils. Then the subjective probability $p(A)$ equals the price (specified in utils) that the decision maker is willing to pay to receive an extra util contingent on the event $A$. Thus, it is much simpler to construct subjective probability in Anscombe and Aumann’s framework than in Savage’s more general counterpart. However, Anscombe and Aumann’s approach is not innocuous: it takes objective probabilities for granted and requires a more complex structure for objects of choice. Thus, Savage’s theory is commonly viewed as a more solid foundation for subjective probability.6

**REFINING SUBJECTIVE PROBABILITIES:**

**DE FINETTI’S THEOREM, BAYES’S LAW, MARKET SELECTION**

Both Theorems 3.1 and 3.2 above seek to explain why a rational person should have a probabilistic belief and how he or she should use it in decision making. These models put absolutely no restrictions on what personal beliefs should be, as long as they comply with the mathematical definition of probability. Thus, in applications it may be desirable to derive additional structure for subjective probability.

Consider first a setting where uncertainty is resolved in a state space

$$S = \{0, 1\}^N = \{0, 1\} \times \{0, 1\} \times \cdots$$

via a sequence of identical experiments. For example, let this sequence be produced by flipping a coin, by shooting basketball free throws, or by screening random votes at a poll. Objective statisticians commonly model such experiments as independent with unknown probability of success $\pi$ in each trial. Yet for the decision maker in Theorems 3.1 and 3.2, statistical independence may be counterintuitive. For instance, he or she may strongly prefer to bet on a free throw after 10 have

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6Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003) show that, in principle, one can reformulate any model in Anscombe and Aumann’s framework without the use of objective randomization.
been made rather than after 10 have been missed. Thus, the decision maker may adjust his or her beliefs to experience in a way that violates independence.

Yet a weaker condition called exchangeability (or symmetric dependence) is still intuitive for subjective probabilities $p(\cdot)$ in this setting. This condition requires that beliefs are unaffected by the labeling of experiments. Formally, for any fixed sequence or zeros and ones $a_1, \ldots, a_n$ and for any permutation $\theta$ of the set $\{1, \ldots, n\}$,

$$p(\{s : s_1 = a_1, \ldots, s_n = a_n\}) = p(\{s : s_1 = a_{\theta(1)}, \ldots, s_n = a_{\theta(n)}\}).$$

For example, the events “tails, then heads twice” and “heads, tails, heads again” in three initial coin tosses should have the same probabilities (not necessarily equal to $\frac{1}{8}$). Despite its innocuous appearance, exchangeability has some striking implications established by de Finetti.

**Theorem 3.3.** If $p$ is exchangeable, then there is a unique probability measure $\mu$ on $[0, 1]$ such that for any sequence $a_1, \ldots, a_n$ that has $k$ ones and $n-k$ zeros,

$$p(\{s : s_1 = a_1, \ldots, s_n = a_n\}) = \int_0^1 \pi^k (1-\pi)^{n-k} \, d\mu(\pi).$$

This result asserts that the decision maker with exchangeable beliefs behaves as if he or she integrates binomial distributions with respect to some prior distribution on the parameter space $[0, 1]$. De Finetti’s theorem has another important lesson. For each $s \in S$ and $n$, let $\hat{\xi}_n(s) = \frac{s_1 + \cdots + s_n}{n}$ be the empirical frequency of successes in the first $n$ trials, and let $\pi_n(s)$ be the subjective probability of success in $n$th trial conditional on the observation of $s_1, \ldots, s_{n-1}$. Then for an exchangeable $p$,

$$p(\{s : \lim \pi_n(s) = \lim \hat{\xi}_n(s)\}) = 1.$$

Thus, the decision maker believes that with probability one (1) the empirical frequencies $\hat{\xi}_n(s)$ converge, (2) the subjective conditional beliefs $\pi_n(s)$ are well defined and converge, and (3) $\hat{\xi}_n(s)$ and $\pi_n(s)$ have the same limits. In other words, the decision maker believes that asymptotically his or her beliefs will coincide with empirical frequencies. Generalizations of de Finetti’s theorem are obtained by Hewitt and Savage (1956) and Diaconis (1977).

Subjective probability in Theorems 3.1 and 3.2, as well as in de Finetti’s theorem, is static. Accordingly, these results say nothing about how personal beliefs should evolve over time as new information becomes available. The standard updating procedure in statistics is Bayes’s law, which asserts that the posterior probability of an event $A$ after an event $B$ has been observed is the ratio of the prior probabilities $p(A \cap B)/p(B)$. To derive this law for subjective probability in a temporal setting, one needs to impose the principle of dynamic consistency. In Savage’s framework, this principle asserts that the decision maker who has ex ante preference $fA \succeq g$ should still prefer $f$ to $g$ after observing $A$ (see Ghirardato, 2002). However, people may violate Bayes’s law and the associated dynamic consistency because of deeper introspection, surprise, or cognitive dissonance. See Epstein (2006) for a model of non-Bayesian updating.

Alternatively, one can use evolutionary arguments to show that eventually people with the most accurate beliefs will dominate the population. For instance, if evolution is performed by markets, asset prices should eventually reflect rational beliefs. This market selection hypothesis, due to Alchian (1950) and Freedman (1953), has been recently confirmed theoretically by Sandroni (2000). In a complete market setting with a common discount factor, Sandroni shows that if there are agents who eventually make accurate predictions, then these agents are exactly those who survive in the long run, and the market prices eventually reflect true objective probabilities.

**Empirical Content of Subjective Probability**

Normative models of subjective probability, such as Theorems 3.1 and 3.2, have many practical limitations. First, it may be hard to establish a one-to-one correspondence between physical actions and Savage-style abstract acts. Indeed, many decisions that people make under uncertainty are not easily formulated in terms of states of the world and outcomes. For example, consider a person who has a choice between jobs in two different cities, say New York
and Los Angeles. It is easy to write a long list of uncertainties—such as salary growth, professional satisfaction, colleagues, housing, neighbors, commute, nightlife, weather—that may be relevant for this choice. Yet it is not clear how to specify a state space \( S \) and a set of outcomes \( X \). The decision maker himself or herself may fail to identify \( S \) and \( X \) which are exhaustive, sufficiently detailed, and independent of each other. Indeed, it seems quite reasonable for the decision maker to admit that he or she cannot foresee all contingencies that may occur in New York or Los Angeles. Even if the decision maker does have some particular \( S \) and \( X \) in mind, then how can the modeler learn about them? In principle, the modeler could take universal \( \hat{S} \) and \( \hat{X} \) that include all states of the world and outcomes that people might possibly have in mind. But then what is the act that describes a particular job for a particular person in the universal framework?

On the other hand, there are many choice problems that can be embedded naturally in some Savage-style framework. For example, a number of hypothetical and real experiments in decision theory involve bets on the color of a ball drawn randomly from an urn. In this context, states are colors and outcomes are monetary. Then there is a concern that some acts are pure abstractions that do not correspond to any natural physical actions. This concern is especially grave in Theorems 3.1 and 3.2, where \( S \) must be infinite. Many acts and events in this setting are too complex to have any practical meaning. It is clearly impossible to observe preferences over such acts with any reasonable degree of approximation.

The inherent complexity of the construction of subjective probability in Savage’s framework is a normative concern as well. The reliance on purely abstract objects that cannot be used in any physical or mental experiment weakens the normative power of Savage’s theory. Instead, one might derive subjective probability on a finite state space, as in Anscombe and Aumann (1963), Gul (1993), or Abdellaoui and Wakker (2005). However, the gain in simplicity is illusory because these models impose an additional structure on the set of outcomes and less transparent axioms on preference.

Indecision, Framing, and Regret

The first postulate of the theory of subjective probability is that preferences are complete and transitive. These assumptions are often violated by experimental evidence. One explanation is that people often fail to produce a compelling argument in favor of any of the available alternatives. If they are forced to make a choice while being in this state of indecision, the response may be random and easily manipulated by the experimenter’s presentation of different alternatives. For example, consider a hypothetical situation when an outbreak of a rare Asian disease may kill up to 600 victims and there are several programs to combat the disease:

- program \( f \) will save 200 victims;
- program \( g \) with probability \( \frac{1}{3} \), will save all victims and with probability \( \frac{2}{3} \), will save none of them;
- if program \( f' \) is adopted, then 400 victims will die;
- if program \( g' \) is adopted, then with probability \( \frac{1}{3} \), none of the victims will die and with probability \( \frac{2}{3} \), all victims will die.

Kahneman and Tversky (1983) report an experiment where 72% of decision makers prefer \( f \) to \( g \), and 78% prefer \( g' \) to \( f' \). Yet in real terms, programs \( f \) and \( g \) are indistinguishable from \( f' \) and \( g' \), respectively.

Indecision may also lead to intransitivity of preference. For example, given a slightly bent coin and three events

\[
A = \{ \text{The next 101 flips will give at least 40 heads} \},
\]

\[
B = \{ \text{The next 100 flips will give at least 41 heads} \}, \text{ and}
\]

\[
C = \{ \text{The next 1,000 flips will give at least 460 heads} \}.
\]

A person may have betting preferences \( A \sim_0 C \) and \( B \sim_0 C \) due to indecision but firmly prefer to bet on \( A \) rather than on \( B \). Another possible explanation for intransitivity among acts is regret. Loomes, Starmer, and Sugden (1991) support this explanation with experimental evidence.

Allais’s Paradox

In settings originally studied by Allais (1953), all events have explicit numerical probabilities,
but preferences cannot be represented by expected utility. For example, Kahneman and Tversky (1979) report that a majority of subjects in their experiments prefer to get $3,000 for sure rather than $4,000 with probability 0.8, but also prefer to get $4,000 with probability 0.2 rather than $3,000 with probability 0.25. These choices constitute a violation of expected utility for any index $u$: $u(3000) > 0.8u(4000)$ but $0.25u(3000) < 0.2u(4000)$. Thus, it is hard to believe that expected utility holds in more general settings where objective probabilities are not given. Is it possible to separate subjective probability from the expected utility criterion?

Machina and Schmeidler (1992) propose an elegant solution. Their model of probabilistic sophistication portrays a person who ranks acts in two stages: first, the person uses subjective probabilities to translate each act into a lottery—a distribution over outcomes—and then he or she ranks the induced lotteries via a risk preference, which need not be represented by expected utility. Formally, a probabilistically sophisticated person, who has a belief $p$, reduces every act $f$ to a distribution $l_p(f)$ assigning a probability $p\{s : f(s) = x\}$ to each outcome $x$. Then he or she evaluates the act $f$ via a utility function

$$U(f) = V(l_p(f)),$$

where $V$ represents the risk preference over all relevant lotteries. To accommodate probabilistic sophistication, Machina and Schmeidler (1992) relax the sure-thing principle and require roughly that only the betting preference can be conditioned on any event $A$ independently of outcomes that are obtained if $A$ does not occur. This separability of preference need not hold when acts more complex than bets are conditioned on $A$.

Ellsberg’s Paradox

Another famous paradox, due to Ellsberg (1961), illustrates the empirical importance of the Knightian distinction between risk, which can be represented by numerical probabilities, and ambiguity, which cannot.\(^7\) In particular, this paradox arises when a person is told that (1) a ball will be drawn randomly from an urn that contains balls of three possible colors (red, green, and blue), and (2) the probability of drawing a red ball is $\frac{1}{3}$. Then the typical preference is to bet on the event $\{R\}$ rather than on the event $\{B\}$ because the probability of $\{R\}$ is known to be $\frac{1}{3}$, while the probability of $\{B\}$ is not known precisely and lies between 0 and $\frac{1}{2}$. Analogously, it is typical to bet on $\{B,G\}$ rather than on $\{R,G\}$. This betting preference cannot be represented by any subjective probability measure $p$ because the inequalities $p(\{R\}) > p(\{B\})$ and $p(\{B\}) + p(\{G\}) > p(\{R\}) + p(\{G\})$ are inconsistent with the additivity of $p$. What could be the meaning of subjective probability in this case?

To accommodate Ellsberg-type behavior, one can use a utility representation called epsilon contamination:

$$U(f) = \epsilon \min_{q \in \Delta} \int_S u(f(s)) dq + (1 - \epsilon) \int_S u(f(s)) dp. \quad (3.6)$$

Here, the set $\Delta$ is given exogenously: it consists of all probabilistic scenarios on $S$ that are consistent with the available objective evidence. For example, in the Ellsberg paradox $\Delta = \{q : q(\{R\}) = \frac{1}{3}\}$. The decision maker, as portrayed by (3.6), evaluates every act $f$ via an $\epsilon$-mixture of the most unfavorable scenario in $\Delta$ and the subjective probability $p$. Note that the weight $1 - \epsilon$ can be interpreted as a degree of confidence that the decision maker has in his or her belief $p$. Kopylov (2006) characterizes epsilon contamination by relaxing Anscombe and Aumann’s postulates of the subjective expected utility theory. Note also that representation (3.6) is a special case of the multiple priors model due to Gilboa and Schmeidler (1989). However, unlike this more general model, epsilon contamination specifies a unique and additive probability measure that underlies choice among uncertain prospects. Thus it suggests that subjective probability can be meaningful even for people who distinguish between risk and ambiguity and behave accordingly.

**Summary**

To conclude, we evaluate the concept of subjective probability via the following three criteria

\(^7\)Knight (1921) refers to *uncertainty* rather than *ambiguity*. Modern literature widely uses Ellsberg’s terminology, where uncertainty is comprehensive and includes both risk and ambiguity.
due to Salmon (1966): (1) coherence (or admissibility), so that probability should comply with Kolmogorov’s definition; (2) ascertainability, so that at least in principle, it should be possible to find out values of probabilities; and (3) applicability, so that it should be clear how probability relates to the objective world and how it should be used in decision making.

Subjective probability derived in Theorems 3.1 and 3.2 is coherent but does not fully satisfy the other two criteria. First, to find out the values of subjective probability, one needs to find out a preference relation $\succeq$ that satisfies a suitable list of rationality conditions. Even in principle, such a preference is hard to come by. Next, both Theorems 3.1 and 3.2 relate subjective probabilities to decision making rather than to any observations of the physical world.

Thus, there is a growing body of literature that (1) derives subjective probability from weaker primitives and (2) relates subjective probability with objective evidence. The full impact of this literature remains to be seen.

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