The Geometry Behind Poincaré’s Conventionalism

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A Duhemian global reading:

1. The metric of space is underdetermined by our a priori and empirical evidence.
2. Given such underdetermination, only convention can be used to decide on a metric.
3. Matters of convention are not matters of fact.

So, the metric of space is not a matter of fact, but of convention.

Friedman 1999 points out that this argument proves too much:

i. Premise 1 is true of all sentences in any total physical theory, not just the sentences of physical geometry.
ii. Premise 2 seems to make every sentence in our total theory of space (including claims of physics about temperature, gravitation, etc.) conventional.
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- Ben-Menahem grounds Poincaré’s conventionalism on the existence of models of Non-Euclidean geometry.

- But these models predate Poincaré. Why did conventionalism have to wait for Poincaré?

- Russell 1897: Conventionalism arises from a certain way of interpreting these models.
Russell: conventionalism depends on the resolution of a debate between Klein (and Poincaré) and Beltrami

Since these systems are all obtained from a Euclidean plane, by a mere alteration in the definition of distance, Cayley and Klein [though not Beltrami] tend to regard the whole question as one, not of the nature of space, but of the definition of distance. Since this definition, on their view, is perfectly arbitrary, the philosophical problem vanishes…, and the only problem that remains is one of convention and mathematical convenience. This view has been forcefully expressed by Poincaré*: "What ought one to think," he says, "of this question: Is the Euclidean geometry true? The question is nonsense." Geometrical axioms, according to him, are mere conventions: they are "definitions in disguise." [Essay, §33]

* ["Non-Euclidean Geometries" (1891), repr. Science and Hypothesis].
Russell’s “Local” Reading

• Russell ultimately rejects the Klein/Cayley/Poincaré point of view, and thus rejects conventionalism:
  – "The projective geometer…when he introduces the notion of distance, he defines it, in the only way projective principles allow him to define it, as a relation between four points." (§37)
  – But distance is a relation between two points only.
  – "Beltrami remains justified as against Klein" (§33).

• Similar readings: Reichenbach 1920, Zahar 1997
Overview of the talk

Section I: Early Russell versus Poincaré

• I explain Russell’s objection to Poincaré, its philosophical motivations, and why it fails as a reading of Poincaré.

Section II: From Fuchsian Functions to Coventionalism

• I argue that there is a successful modified Russellian reading.
• There is present in Poincaré an argument for conventionalism from the possibility of alternative definitions of distance within pure mathematics.
• This argument does not derive from the underdetermination of physical geometry by experience, but by the application of one mathematical theory to another.
To understand Russell’s objection, we need to understand the different routes Beltrami and Klein took to their model:

Beltrami used methods from differential geometry, and (in his 1868 *Saggio*) constructed a 2-d model in Euclidean 3-space.

Klein ("On the so-called Non-Euclidean Geometry" [1871]), however, constructed the model within complex projective 2-space, using projective geometry.
Beltrami begins by inducing on a plane the metric of the sphere with radius R.

We project the southern hemisphere of a sphere onto a plane tangent to its south pole using the center C of the sphere as the point of projection.

Great circles on the sphere are projected onto lines in the plane.
The equation for the distance function $ds^2$ on the plane depends on $u$, $v$, and $R^2$, and is therefore meaningful even if $R$ is replaced with $R\sqrt{-1}$.

Beltrami then showed that replacing $R$ with $R\sqrt{-1}$ induces on a plane a metric that has constant negative.

The points on the sphere projected from $C$ are now inside a disk.
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Klein’s Construction of the Model

Von Staudt showed how to induce a set of coordinates on the points on a projective line by repeated application of a purely projective construction (viz. the quadrilateral construction), which picks out four tuples of points that always have the same cross ratio, with two points assigned the values $n$ and $\infty$. 
\[ CR(P, A, B, Q) = \left( \frac{PA}{PB} \right) \left( \frac{QB}{QA} \right) \]

\[ CR(P, A, B, Q) \cdot CR(P, B, C, Q) = CR(P, A, C, Q) \]

\[ d(A, B) = c[\log CR(P, A, B, Q)] \]
So we could define distance entirely in projective terms -- if only we had a systematic way to pick out two arbitrary points, \( n \) and \( \infty \), on a line.

But every line intersects a conic in the complex projective plane in two points, conics determine cross ratios, and conicity can be defined purely projectively.

So, we can define distance projectively if we just pick an arbitrary conic in the plane.
Indeed Cayley ("Sixth Memoir on Quantics") showed how to construct the standard Euclidean distance function in this way, if a certain kind of conic is picked out.

Klein's idea was that different kinds of metric geometries would arise if the conic were chosen differently.
If the fundamental conic is imaginary, then we have spherical geometry (spaces of constant positive curvature).

If the fundamental conic is imaginary and degenerates into a point pair, then we have Euclidean geometry (spaces of 0 curvature).

If the fundamental conic is real, not ruled, and it encloses us, then we have hyperbolic geometry (spaces of constant negative curvature).
In the third case, \( c \) becomes \(-1/2\), and we get the standard Klein-Beltrami model of hyperbolic geometry.

\[
d(A,B) = \frac{1}{2}[\log CR(P,A,B,Q)]
\]

In group theoretic terms, hyperbolic properties are invariant under coordinate transformations that are projective (and so preserve cross ratio) and leave unchanged this fundamental real conic.
The Poincaré disk model can be derived from the Beltrami-Klein model.

Take the Beltrami disk as the equatorial disk of a sphere.
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Take the Beltrami disk as the equatorial disk of a sphere.

Project the disk orthographically down onto the southern hemisphere.
Then project the bottom hemisphere onto the plane from the north pole of the sphere.
Then project the bottom hemisphere onto the plane from the north pole of the sphere.
Russell’s Criticism

Russell reconstructs Poincaré’s argument in the following way:

1. The Cayley-Klein metric defines $d(A,B)$ in terms of the cross ratio of A and B, and two other arbitrarily chosen points P and Q.
2. The procedure for picking arbitrary points P, Q on a line correspond to the three geometries of constant curvature. So, the choice among the three geometries of constant curvature is arbitrary.

The idea: the three metric geometries are just different ways of describing the same underlying complex, projective plane.

But Russell rejects premise 1.
"Distance, in the ordinary sense, remains a relation between two points, not between four; and it is the failure to perceive that the projective sense differs from, and cannot supersede, the ordinary sense, which has given rise to the views of Klein and Poincaré." (Russell 1897, §37)

(Russell maintains that distance is a relation between four points because he wants geometry to play a transcendental role: to provide the form of externality.)

But...Poincaré 1898: “To arrive at the notion of length by regarding it merely as a particular case of the anharmonic ratio is an artificial and repugnant detour.”
Section II: From Fuchsian Functions to Coventionalism

- Note: Russell’s reading does not depend on any facts about *physical geometry*, the underdetermination of total theories by physical evidence, or Poincaré’s famous sphere argument.
- It depends on the fact that *distance* can be defined by applying one area of mathematics to another.
Poincare’s Earliest Conventionalist Arguments Predate the Sphere Model

• Poincaré 1887: “If geometry is nothing but the study of a group, one may say that the truth of the geometry of Euclid is not incompatible with the truth of the geometry of Lobachevsky, for the existence of a group is not incompatible with that of another group.”

• Poincaré 1891: Poincaré gives the half space model of hyperbolic 3-space using a “dictionary.” He concludes “Nothing remains then of the objection [that there may be a hidden contradiction in Lobachevskian geometry] above formulated. This is not all. Lobachevski's geometry, susceptible of a concrete interpretation, ceases to be a vain logical exercise and is capable of applications; I have not the time to speak here of these applications, nor of the aid that Klein and I have gotten from them for the integration of linear differential equations.”
Poincare on Fuchsian Functions

• Poincaré’s first application of hyperbolic geometry was in the theory of Fuchsian functions.

• Fuchsian functions are a generalization of elliptic functions.
Elliptic Functions

Elliptic integrals are integrals of the form

$$\int R[t, \sqrt{p(t)}] \, dt$$

where $R$ is a rational function and $p(t)$ is a polynomial of degree 3 or 4. An example of such an integral is the lemniscatic integral

$$\int_{0}^{x} \frac{dt}{\sqrt{1-t^4}}$$

which gives the arc length of the lemniscate of Bernoulli, whose Cartesian equation is

$$\left(x^2 + y^2\right)^2 = x^2 - y^2$$
\[
\int_{0}^{x} \frac{dt}{\sqrt{1-t^4}}
\]

\[
(x^2 + y^2)^2 = x^2 - y^2
\]
Elliptic Integrals

- These functions have been studied since Leibniz.
- They cannot be expressed in terms of elementary functions, which frustrated the intuitive natural idea (beginning with Leibniz himself) that the solution of every integration problem should be expressed in terms of elementary functions.
- A breakthrough came around 1800 when Gauss and others studied their inverses.
- The inverse of an elliptic integral is called an elliptic function.
- For example, the inverse of the leminscatic integral Gauss called a "lemniscatic sine function," on analogy with the sine function.
Sine function

\[ f(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} \]

\[ f^{-1}(x) = \sin(x) \]

Lemniscatic Sine

\[ \int_0^x \frac{dt}{\sqrt{1-t^4}} \]

\[ f^{-1}(x) = sl(x) \]
\[ \sin(x) \text{ and } sl(x) \]

- The integral
  \[ f(x) = \int_{0}^{x} \frac{dt}{\sqrt{1-t^2}} \]
  gives the formula for the arc length of a circle.

- The equation for the circle can be parameterized so that
  \[ x = \sin(t) \text{ and } y = \cos(t) = \sin'(t) \]

- Just as \( \sin(t) \) can be used to parameterize the equation of a circle, so too can an elliptic function be used to parameterize the equation of certain curves. That is, if \( f(x) \) is an elliptic integral, then there are curves that can be parameterized so that
  \[ x = f^{-1}(u) \quad y = f^{-1'}(u) \]

- These are called elliptic curves.
Sin functions are periodic. They are invariant under substitutions

\[ x \mapsto x + 2n\pi \]

Gauss noticed that the lemniscatic sine function is doubly periodic in the complex plane.

\[ f^{-1}(w) = f^{-1}(w + m\omega_1 + n\omega_2) \]

for complex numbers \( \omega_1 \) and \( \omega_2 \) and any integers \( m, n \).
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This can be represented geometrically by a plane tiled by parallelograms.

The mapping of *-points to points on the elliptic curve will be unaffected by

\[ z \mapsto z + m\omega_1 + n\omega_2 \]

These are just rigid translations of the plane that keep the tiling intact. Each element of the group corresponds to a tile.
Fuchsian Functions

• Elliptic functions parameterize every elliptic curve.
• But could this be generalized to other algebraic curves?
• In 1880, Poincaré studied a generalization proposed by Lazarus Fuchs, who (on analogy with elliptic integrals) studied the inversions of integrals of quotients of linear differential equations.
• Instead of being doubly periodic, Fuch’s functions are invariant under linear fractional transformations

\[ z \mapsto \frac{az + b}{cz + d} \]

• But do they exist? Is the inversion single-valued?
In his Prize Essay of 1880, Poincaré discovered that the inversion is only well-defined within a disk around the origin.

Further, F is defined within a region only if the tessellation of the complex plane by F never overlaps itself.

To understand this, one needs to understand the geometry of the tessellation.

In 1880 he notes that the tessellation has the form of the Poincaré disk.

He projects it stereographically and orthogonally onto the Beltrami-Klein disk model.

He notices that the disk is a model of hyperbolic geometry.

He then uses facts about hyperbolic geometry to prove facts about this tessellation (namely, that it does not overlap).
The geometry on the tesselated complex plane induced by elliptic functions is Euclidean and easy to understand.
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The tesselations induced by Fuchsian functions are much more complicated. They become tractable when seen as a model of hyperbolic geometry.

The elements of the group of automorphic functions that leave solutions to the Fuchsian functions invariant are tiles.
Poincaré (along with Klein) proved later in the 1880s that every algebraic curve could be paramaterized by Fuchsian functions and their derivatives.
The Argument for Conventionalism from Pure Geometrical Undeterdetermination

Improved Russellian Reading:

1. A class of isometries on the plane can be defined in terms of a (Fuchsian or elliptic) function.

2. The choice of a Fuchsian or elliptic function determines whether the plane is Euclidean or Non-Euclidean.

So, the choice among Euclidean or Non-Euclidean geometry is arbitrary, and depends on what kind of function one has in mind.

Poincaré’s application of hyperbolic geometry has the same relevant feature of the Cayley-Klein metric: he defines a class of isometries of the complex plane by applying one area of mathematics (theory of complex functions) to another.
Conclusions

There is an argument for conventionalism about geometry present in Poincaré’s writings that does not depend on Duhemian underdetermination of physical theory.

This argument depends on the possibility of applying pure geometry to other areas of pure mathematics. These applications allow for definitions of distance. Since these definitions are not unique, there is no fact of the matter about which notion of distance is the correct one.

These kinds of applications were given originally and brilliantly in Poincaré’s mathematical work in the 1880s.